Published by Institute of Physics Publishing for SISSA
Received: *May 11, 2007*
Accepted: *June 14, 2007*
Published: *July 2, 2007*

On purely transmitting defects in affine Toda field theory

Edward Corrigan
Department of Mathematics University of York
York YO10 5DD, U.K.
E-mail: ec9@york.ac.uk

Cristina Zambon
Laboratoire de Physique Théorique et Modélisation,
Université de Cergy-Pontoise (CNRS UMR 8089), Saint-Martin 2,
2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
E-mail: cristina.zambon@ptm.u-cergy.fr

Abstract: Affine Toda field theories with a purely transmitting integrable defect are considered and the model based on a_2 is analysed in detail. After providing a complete characterization of the problem in a classical framework, a suitable quantum transmission matrix, able to describe the interaction between an integrable defect and solitons, is found. Two independent paths are taken to reach the result. One is an investigation of the triangle equations using the S-matrix for the imaginary coupling bulk affine Toda field theories proposed by Hollowood, and the other uses a functional integral approach together with a bootstrap procedure. Evidence to support the results is collected in various ways: for instance, through the calculation of the transmission factors for the lightest breathers. While previous discoveries within the sine-Gordon model motivated this study, there are several new phenomena displayed in the a_2 model including intriguing disparities between the classical and the quantum pictures. For example, in the quantum framework, for a specific range of the coupling constant that excludes a neighbourhood of the classical limit, there is an unstable bound state.

Keywords: Integrable Field Theories, Boundary Quantum Field Theory, Exact S-Matrix.

Contents

1. Introduction

More than a decade ago, Delfino, Mussardo and Simonetti [1] kindled interest in examining defects in integrable quantum field theories and since then some progress has been made in various directions although there remain many open problems. It is not the purpose of this article to review all the subsequent developments but a few remarks are in order. The field theories to be discussed in this paper are non-conformal and describe when quantised a collection of massive particles. Within a free massive field theory a defect, for example a defect (or impurity) of δ-function type, will be accompanied by both transmission and reflection, and perhaps extra bound states specifically associated with the defect. However, at least at a classical level, a δ-function defect within a nonlinear integrable model will destroy integrability. Also within a quantum field theory containing a defect, the algebraic constraints to be satisfied by the bulk S-matrix, and the reflection and transmission factors, as described in [1, 2] are extremely stringent, and may only be satisfied with non-zero reflection and transmission provided the bulk S-matrix is a constant independent of rapidity. Later on, an alternative scheme was developed by Mintchev, Ragoucy and Sorba [3], by requiring the reflection and transmission matrices to satisfy a different algebra. Within this scheme the S-matrix need not be trivial in the presence of non-zero reflection and transmission. For particular quantum field theories - such as the sine-Gordon model, or more generally any of the affine Toda field theories - with a δ-function defect, it remains to be seen which of these schemes, if indeed either of them, might turn out to be correct.

On the other hand, one might ask a different question and explore defects that are known to be integrable within the classical field theory, meaning that they do not destroy classical integrability, and subsequently study their role within the corresponding quantum field theory. This was the approach taken in [4] and then applied to a subset of the affine Toda field theories in [5]. For nonlinear models, integrable defects, such as those described in [4], require discontinuities in the fields at the location of a defect (rather than discontinuities in their derivatives, which would be typical of a δ-function discontinuity in a nonlinear wave-equation), with specified defect conditions relating the fields on either side of the defect. For this reason, they are sometimes called 'jump'-defects to emphasise the fact the fields are themselves discontinuous. Interestingly, the defect conditions turn out to be reminiscent of Bäcklund transformations 'frozen' at the site of the defect. For a recent treatment of these defects and extensions to other models see [6, 7]. Typically, these defects are purely transmitting from a classical point of view and, for example in the sine-Gordon model, solitons will pass through the defect - though not unscathed; generally they will be delayed and might, depending on the precise circumstances, be converted to an antisoliton, or be absorbed. Integrable defects studied so far also explicitly break some or all of the discrete symmetries usually enjoyed by the bulk theories, the main examples being parity and time-reversal. This fact implies that solitons travelling from $x < 0$ towards $x > 0$ ('left to right') will be affected by the defect in a different manner to those travelling in the opposite direction.

In a recent article [8], devoted to integrable, purely transmitting defects within the sine-Gordon model, it was shown how the classical defects, introduced in [4], may be incorporated within the associated quantum field theory. In particular, it was demonstrated how the transmission matrix discovered originally by Konik and LeClair [9] naturally describes the behaviour of solitons passing through a defect, with the quantum versions of the soliton-defect scattering properties matching very closely the classical features. More precisely, there are two transmission matrices, one of them labelled by even integers and the other labelled by odd integers. Alternatively, these may be described equivalently by the roots and weights of a_1: one of the transmission matrices (with even labels) being labelled naturally by roots (or integer spin weights), the other being labelled by the weights of the other representations (those of half odd integer spin). It is natural to regard the transmission matrix labeled by roots as being unitary (since the sine-Gordon model is a unitary quantum field theory), but then the transmission matrix labelled by the other weights turns out not to be unitary. In fact, the states corresponding to the defects described by the latter are unstable soliton-defect bound states. The appearance in this context of unstable states is an interesting new feature of the sine-Gordon model. It was also shown how it is possible consistently to allow the classical defects to move and scatter among themselves. Yet, it remains to be seen how this feature will be realised in the quantum field theory. Finally,

JHEP07(2007)001

although convincing non-perturbative arguments were provided for the soliton transmission matrices described in [8], it was also shown that breather transmission matrices are particularly simple and are, at least in principle, amenable to perturbative calculations.

It is natural to ask if any of these features of integrable defects will emerge in the imaginary coupling quantum affine Toda field theories based on data associated with other algebras. The sine-Gordon model is the only unitary model within this class of quantum field theories and yet it was pointed out by Hollowood [10] that the classical complex solitons found within a general affine Toda field theory have real energy and momentum, and moreover their scattering might be described by non-unitary S-matrices satisfying bootstrap and crossing relations [11]. An assumption made by Hollowood concerned the spectrum of quantum solitons: these are supposed to be multiplets corresponding to the fundamental representations of the Lie algebra whose data is used to define a particular affine Toda field theory (for early references, see [12]). However, a curious feature of the associated classical field theory is that, apart from the models based on a_1 and a_2, the spectrum of classical static solitons is actually different and, in almost all cases, most of the solutions that should have topological charges corresponding to weights within a fundamental representation are actually missing; as has been noted by McGhee [13]. Alternative methods of constructing solutions [14] have not so far revealed the absentees. Presumably, the extra states in these quantum models are dynamically generated although no detailed mechanism has been proposed to achieve this. It is tempting to speculate that defects may have something to do with the story and this idea has provided a partial motivation for this paper.

2. Jump-defects in the classical a_r affine Toda field theories

This article will focus on a subset of affine Toda field theories, namely those associated with the root data of the Lie algebras a_r, and in particular of a_2. Apart from having the most symmetrical root/weight systems, these are the models for which classically integrable defects have been described in detail, whose complex solitons are easy to describe, and whose full set of S-matrices are relatively easy to calculate using the bootstrap.

In the bulk, $-\infty < x < \infty$, an affine Toda field theory corresponding to the root data of the Lie algebra a_r is described conveniently by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\cdot\partial^\mu\phi - \frac{m^2}{\beta^2}\sum_{j=0}^{r}(e^{\beta\alpha_j\cdot\phi}-1), \tag{2.1}$$

where m and β are constants, and r is the rank of the algebra. The vectors α_j with $j = 1,\dots,r$ are simple roots (with the convention $|\alpha_j|^2 = 2$), and α_0 is the lowest root, defined by

$$\alpha_0 = -\sum_{j=1}^{r}\alpha_j.$$

The field $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ takes values in the r-dimensional Euclidean space spanned by the simple roots $\{\alpha_j\}$. The extra root α_0 distinguishes between the massive affine and the massless non-affine Toda field theories. The massive affine theories are integrable, possessing infinitely many conserved charges, a Lax pair representation, and many other interesting properties, both classically and in the quantum domain. The simplest choice ($r = 1$) coincides with the sinh-Gordon model. For further details concerning the affine Toda field theories, see [12, 15] and the review [16], where further references can be found.

After quantisation, provided the coupling constant β is real, and the fields are restricted to be real, the a_r affine Toda field theory describes r interacting scalars, also known as fundamental Toda particles, whose classical mass parameters are given by

$$m_a = 2\,m \sin\left(\frac{\pi a}{h}\right), \quad a = 1, 2 \ldots, r, \tag{2.2}$$

where $h = r + 1$ is the Coxeter number of the algebra. On the other hand, if the fields are permitted to be complex each affine Toda field theory possesses classical 'soliton' solutions [10]. Conventionally, complex affine Toda field theories are described by the Lagrangian density (2.1) in which the coupling constant β is replaced with $i\beta$. Once complex fields are allowed it is clear that the potential appearing in the Lagrangian density (2.1) vanishes whenever the field ϕ is constant and equal to

$$\phi = \frac{2\pi\, w}{\beta} \quad \text{with} \quad \alpha_j \cdot w \in \mathbf{Z}, \quad \text{i.e.} \quad w \in \Lambda_W(a_r), \tag{2.3}$$

where $\Lambda_W(a_r)$ is the weight lattice of the Lie algebra a_r. These constant field configurations have zero energy and correspond to stationary points of the affine Toda potential. Soliton solutions smoothly interpolate between these vacuum configurations as x runs from $-\infty$ to ∞. It is natural to define the 'topological charges' characterizing such solutions as follows:

$$Q = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dx\, \partial_x\phi = \frac{\beta}{2\pi}\left[\phi(\infty, t) - \phi(-\infty, t)\right], \tag{2.4}$$

and these lie in the weight lattice $\Lambda_W(a_r)$. Assuming $\phi(-\infty, t) = 0$, static solitons may be found for which $\phi(\infty, t)$ lies in a subset of the weight lattice. In particular, there are static solutions corresponding to weights within each of the representations with highest weight $w_a,\ a = 1, \ldots, r$, satisfying

$$\alpha_i \cdot w_a = \delta_{ia}, \quad i, a = 1, \ldots, r. \tag{2.5}$$

Explicitly boosted solutions of this type that correspond to the representation labelled by a have the form

$$\phi^{(a)} = \frac{m^2 i}{\beta}\sum_{j=0}^{r}\alpha_j \ln\left(1 + E_a\, \omega^{aj}\right), \quad E_a = e^{a_a x - b_a t + \xi_a}, \quad \omega = e^{2\pi i/h}, \tag{2.6}$$

where $(a_a, b_a) = m_a\,(\cosh\theta, \sinh\theta)$, ξ_a is a complex parameter, and θ is the soliton rapidity. Despite the solutions (2.6) being complex, Hollowood [10] showed their total energy and momentum is actually real and requires masses for static single solitons proportional to the mass parameters of the real scalar theory. These are given by

$$M_a = \frac{2\,h\,m_a}{\beta^2}, \quad a = 1, 2 \ldots, r. \tag{2.7}$$

Moreover, for each $a = 1, \ldots, r$ there are several solitons whose topological charges lie in the set of weights of the fundamental $a^{\rm th}$ representation of a_r [13]. However, apart from the two extreme cases, $a = 1$ and $a = r$, not every weight belonging to one of the other representations corresponds to a static soliton. The number of possible charges for the representation with label a is exactly equal to the greatest common divisor of a and h, the relevant weights being orbits of the Coxeter element, and explicit expressions for them may be found in [13]. The parameter ξ_a is almost arbitrary but clearly has to be chosen so that there are no singularities in the solution as x, t vary; shifting ξ_a by $2\pi i a/h$ changes the topological charge. For the two extreme representations (with $a = 1$ or $a = r$), it is clear repeated use of this translation processes the charges through the full set of weights.

The affine Toda field theories (2.1) based on a_r generalize the sinh-Gordon model and the primary purpose of this article is to extend the techniques and results of recent work devoted to the sine-Gordon model [8] to investigate the manner in which an integrable discontinuity, or 'jump' defect, can be accommodated within the quantum field theory associated with a more general class of field theories. From a purely classical perspective, the defects have been described before [5]. However, for completeness the main features will be reviewed here together with some additional observations.

There are several types of integrable defect for a_r affine Toda field theory and the distinctions between them are explained in [5]. To maintain clarity, most of the calculations will relate to a specific choice of defect with comments on the other possiblities relegated to the last section. Bearing this in mind, a single defect located at $x = 0$ may be described by the following modified Lagrangian density

$$\mathcal{L}_d = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)\left(\frac{1}{2}(\phi\cdot E\partial_t\phi + \phi\cdot D\partial_t\psi - \partial_t\phi\cdot D\psi + \psi\cdot E\partial_t\psi) - \mathcal{B}(\phi,\psi)\right), \tag{2.8}$$

where E is an antisymmetric matrix, $D = 1 - E$,

$$\mathcal{L}_\phi = \frac{1}{2}\,\partial_\mu\phi\cdot\partial^\mu\phi + \frac{m^2}{\beta^2}\sum_{j=0}^{r}(e^{i\beta\alpha_j\cdot\phi} - 1), \tag{2.9}$$

and

$$\mathcal{B} = -\frac{m}{\beta^2}\sum_{j=0}^{r}\left(\sigma\, e^{i\beta\alpha_j\cdot(D^T\phi + D\psi)/2} + \frac{1}{\sigma}\, e^{i\beta\alpha_j\cdot D(\phi-\psi)/2}\right). \tag{2.10}$$

Here, ϕ and ψ are the fields on the left and on the right of the defect, respectively, and σ is the defect parameter. The matrix D satisfies the following constraints

$$\alpha_k\cdot D\alpha_j = \begin{cases} \ \ 2 \ k = j, \\ -2 \ k = \pi(j), \\ \ \ 0 \ \text{otherwise}, \end{cases} \qquad D + D^T = 2, \tag{2.11}$$

where $\pi(j)$ indicates a permutation of the simple roots. Choosing the 'clockwise' cyclic permutation,

$$\alpha_{\pi(j)} = \alpha_{j-1},\ j = 1, \ldots, r, \quad \alpha_{\pi(0)} = \alpha_r,$$

the set of constraints (2.11) is satisfied by the choice,

$$D = 2\sum_{a=1}^{r} w_a\,(w_a - w_{a+1})^T\,, \tag{2.12}$$

where the vectors $w_a,\ a = 1, \ldots, r$ are the fundamental highest weights of the Lie algebra a_r, with the added convention $w_0 \equiv w_{r+1} = 0$. Note, the 'anticlockwise' cyclic permutation used in [5] is effected by substituting the matrix (2.12) by its transpose.

Given the modified Lagrangian density (2.8) the corresponding equations of motion and defect conditions are, respectively,

$$\begin{aligned}
\partial^2\phi &= \frac{m^2 i}{\beta}\sum_{j=0}^{r}\alpha_j\, e^{i\beta\alpha_j\cdot\phi} \qquad x<0,\\
\partial^2\psi &= \frac{m^2 i}{\beta}\sum_{j=0}^{r}\alpha_j\, e^{i\beta\alpha_j\cdot\psi} \qquad x>0,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\partial_x\phi - E\partial_t\phi - D\partial_t\psi + \partial_\phi\mathcal{B} = 0 \qquad x=0,\\
\partial_x\psi - D^T\partial_t\phi + E\partial_t\psi - \partial_\psi\mathcal{B} = 0 \qquad x=0.
\end{aligned} \tag{2.14}$$

There are several basic properties of (2.14) that are worth noting. Shifting the fields $\phi,\ \psi$ by roots yields another solution with the same energy and momentum. This is because both the bulk and defect potentials are invariant under the translations

$$\phi \to \phi + 2\pi r/\beta, \quad \psi \to \psi + 2\pi s/\beta, \tag{2.15}$$

where r, s are any two elements of the root lattice. In particular, constant fields

$$(\phi,\psi) = 2\pi(r,s)/\beta \tag{2.16}$$

all have the same energy and momentum despite having a discontinuity at the location of the defect. Writing $\sigma = e^{-\eta}$, the energy-momentum of each of these configurations is

$$(\mathcal{E}_0,\ \mathcal{P}_0) = -\frac{2hm}{\beta^2}(\cosh\eta,\ -\sinh\eta). \tag{2.17}$$

Other constant configurations are possible and, because of the invariance under translations by roots, it is enough to consider configurations $(\phi,\psi) = 2\pi(w_p, w_q)/\beta$, where w_p, w_q are fundamental highest weights. These are the other possible constant solutions to (2.14), with energy-momentum given by

$$(\mathcal{E}_a,\mathcal{P}_a) = -\frac{2hm}{\beta^2}\left[\cosh\left(\eta+\frac{2a\pi i}{h}\right), -\sinh\left(\eta+\frac{2a\pi i}{h}\right)\right],\ a=(p-q) \quad p,q = 1,\ldots,r. \tag{2.18}$$

It is perhaps surprising there is a conserved momentum associated with the defect. However, that this should be so was pointed out in [5], and the expressions given there have been used to calculate the above. The expressions in (2.18) are complex, and that is not in itself a surprise, yet all lie on the same mass shell as (2.17), which is perhaps more surprising.

The essential step in calculating (2.18) relies on the fact that the fundamental weights satisfy:

$$w_j \cdot w_p = C^{-1}_{jp}, \text{ where } \alpha_j \cdot \alpha_p = C_{jp},$$

the latter being the Cartan matrix for a_r (see [17] for some details concerning roots and weights). Note, by using (2.11),

$$\frac{1}{2}\alpha_j \cdot D w_p = (w_j \cdot w_p) - (w_{j+1} \cdot w_p), \quad j = 0, \dots, r,$$

and the explicit form of the inverse Cartan matrix,

$$C^{-1} = \frac{1}{h}\begin{pmatrix} r & r-1 & r-2 & \dots & 1 \\ r-1 & 2(r-1) & 2(r-2) & \dots & 2 \\ r-2 & 2(r-2) & 3(r-2) & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & r \end{pmatrix},$$

a direct calculation reveals

$$\frac{1}{2}\alpha_j \cdot D w_p = \frac{a}{h} \quad j \geqslant p, \qquad \frac{1}{2}\alpha_j \cdot D w_p = -\frac{(h-p)}{h} \quad j < p,$$

independently of the label j. Similarly, $(\alpha_j \cdot D^T w_p)/2$ can be calculated.

The system described by the Lagrangian density (2.8) is neither invariant under parity nor under time reversal. By convention, a soliton with positive rapidity will travel from the left to the right and, at some time, it will meet the defect located at $x = 0$. The soliton ψ emerging on the right will be similar to ϕ, but delayed. It is described by,

$$\psi^{(a)} = \frac{m^2 i}{\beta}\sum_{j=0}^{r} \alpha_j \ln\left(1 + z_a\, E_a\, \omega^{aj}\right). \tag{2.19}$$

The expression for the delay z_a was derived in [5] for the 'anticlockwise' permutation. To obtain the delay for the present situation it is enough to send the a^{th} soliton to the $(h-a)^{\text{th}}$ soliton in the formula appearing in [5]. Therefore the delay is given by

$$z_a = \left(\frac{e^{-(\theta-\eta)} + i\, e^{-i\gamma_a}}{e^{-(\theta-\eta)} + i\, e^{i\gamma_a}}\right), \quad \gamma_a = \frac{\pi\, a}{h}. \tag{2.20}$$

The delay is generally complex with exceptions being self-conjugate solitons, corresponding to $a = h/2$ (with r odd), for which the delay is real. In such cases, the delay is equal to the delay found for the sine-Gordon model [4]:

$$z = \left(\frac{1 + e^{-(\theta-\eta)}}{1 - e^{-(\theta-\eta)}}\right) = \coth\left(\frac{\theta-\eta}{2}\right). \tag{2.21}$$

Note also that the delays experienced by a soliton, labelled a, and its associated antisoliton, labelled $\bar{a} = h - a$, are complex conjugates since $z_{\bar{a}} = \bar{z}_a$. For this reason, solitons and antisolitons are expected to behave differently as they pass a defect.

The argument of the phase of the delay (2.20) is given by

$$\tan(\arg z_a) = -\left(\frac{\sin 2\gamma_a}{e^{-2(\theta-\eta)} + \cos 2\gamma_a}\right), \tag{2.22}$$

implying that the phase shift produced by the defect can vary between zero (as $\theta \to -\infty$) and $-2\gamma_a$ (as $\theta \to \infty$), decreasing if necessary through $-\pi/2$ if $\cos 2\gamma_a < 0$. On the other hand, the boundaries between the different topological charge sectors in terms of the imaginary part of ξ_a (eq(2.6)) are separated by exactly $2\gamma_a$. This means that a soliton might convert to one of the adjacent solitons as it passes the defect provided $\arg z_a$ is sufficiently large. In effect, the defect imposes a rather severe selection rule on the possible topological charges of the emerging soliton. In the quantised theory, it is expected that either the transition matrix has zeroes to reflect this selection rule, or severely suppressed matrix elements to represent tunnelling between classically disconnected configurations. In the sine-Gordon model such an effect would not be noticed because the basic representation includes just two states and transitions between them are always permitted.

The delay (2.20) diverges when

$$\theta = \eta + \frac{i\pi}{2}\left(1 - \frac{2a}{h}\right), \tag{2.23}$$

and, with the exception of self-conjugate solitons having $a = h/2$ (including the sine-Gordon model where $(a,h) = (1,2)$), this implies a soliton with real rapidity cannot be absorbed by a defect. For the sine-Gordon model it was noted already that a classical defect can absorb a soliton and, within the quantum theory, this phenomenon implies the existence of unstable bound states. Once the affine Toda field theories are quantised, however, poles in locations given by (2.23) may correspond to additional states that possess no classical counterpart. The positions of the poles are expected to depend on the coupling and it might be the case that there is a range of couplings for which a bound state exists without the range including the classical limit. It is this fact that suggests that defects may be part of the explanation for the missing solitons in the classical models. It will be demonstrated later that a phenomenon rather like this does actually occur in the a_2 model.

More generally, the delay (2.20) satisfies a classical bootstrap in the sense that when two particles a, b in the real quantum field theory have a bound state $\bar{c}$ the corresponding pole in their S-matrix will occur at rapidities

$$\theta_a = \theta_c - i\bar{U}^b_{ac}, \quad \theta_b = \theta_c + i\bar{U}^a_{bc}, \tag{2.24}$$

and the corresponding delays (2.20) in the complex classical theory satisfy

$$z_a(\theta - i\bar{U}^b_{ac})\, z_b(\theta + i\bar{U}^a_{bc}) = z_{\bar{c}}(\theta). \tag{2.25}$$

This is not difficult to check directly using the a_r coupling data [15].

All these observations, and the experience gained with the sine-Gordon model, suggest the investigation of the corresponding quantum theory should be interesting even in the next simplest a_2 model.

JHEP07(2007)001

3. The fundamental S-matrices for the a_r affine Toda field theories

The S-matrices describing the scattering of solitons in the a_r affine Toda field theory were conjectured by Hollowood [11]. Hollowood's proposal makes use of the R-matrices of the quantum group $U_q(a_r)$, specifically the trigonometric solutions of the Yang-Baxter equation (YBE) initially found by Jimbo [18] (and references therein). The basic assumption asserts that the particles of the a_r affine Toda field theory lie in the r different multiplets corresponding to the r fundamental representations of $U_q(a_r)$. The S-matrix S^{ab} describing the scattering of two particles with rapidities θ_1 and θ_2, lying in the multiplets a and b, respectively, is an interwining map on the two representation spaces V_a and V_b. In other words,

$$S^{ab}(\theta_{12}) : V_a \otimes V_b \to V_b \otimes V_a, \quad \theta_{12} = (\theta_1 - \theta_2), \tag{3.1}$$

and S has the following form

$$S^{ab}(\theta_{12}) = \rho^{ab}(\theta_{12})\, R^{ab}(\theta_{12}), \tag{3.2}$$

where R^{ab} is a $U_q(a_r)$ R-matrix and ρ^{ab} is a scalar function determined by the requirements of 'unitarity', crossing symmetry, analyticity and consistency relations (bootstrap constraints), which a scattering matrix must satisfy [11].

For the purposes of the present article, explicit expressions for the S-matrices are needed. In particular, for the a_r affine Toda field theory the explicit expression of the S-matrix describing the scattering of the solitons in the first representation, namely the matrix S^{11} also referred to as the fundamental scattering matrix, will be provided in this section.

The representation space V_1 of the first multiplet has dimension h and its states are the solitons A^1_j, $j = 1, \ldots, h$. The weights of this representation are conveniently described by [17]

$$l^1_j \equiv l_j = \sum_{l=1}^{r} \frac{(h-l)}{h}\alpha_l - \sum_{l=1}^{j-1} \alpha_l, \quad j = 1, \ldots, h. \tag{3.3}$$

The elements of S^{11} can be described conveniently using the non-commutative Faddeev-Zamolodchikov algebra. Consider A^1_j $(j = 1, \ldots, h)$ to be generators of such an algebra. Then, the non-zero elements of S^{11} represent the following relations processes [11, 20] for $j, k = 1, \ldots, h$ and $\theta_{12} > 0$

$$\begin{aligned} A^1_j(\theta_1)A^1_j(\theta_2) &= S^{11}{}^{jj}_{jj}(\theta_{12})\, A^1_j(\theta_2)A^1_j(\theta_1), \\ A^1_j(\theta_1)A^1_k(\theta_2) &= S^{11}{}^{kj}_{jk}(\theta_{12})\, A^1_k(\theta_2)A^1_j(\theta_1) + S^{11}{}^{jk}_{jk}(\theta_{12})\, A^1_j(\theta_2)A^1_k(\theta_1), \quad j \neq k, \end{aligned} \tag{3.4}$$

with

$$
\begin{aligned}
S^{11}{}^{jj}_{jj}(\theta_{12}) &= \rho^{11}(\theta_{12})\,\left(q\,x_{12} - q^{-1}\,x_{12}^{-1}\right), \\
S^{11}{}^{kj}_{jk}(\theta_{12}) &= \rho^{11}(\theta_{12})\,\left(x_{12} - x_{12}^{-1}\right), \quad k \neq j,
\end{aligned}
$$

$$
S^{11}{}^{jk}_{jk}(\theta_{12}) = \rho^{11}(\theta_{12})\,\left(q - q^{-1}\right) \begin{cases} x_{12}^{(1-2|l|/h)}\,|_{l=j-k<0} \\ \\ x_{12}^{-(1-2|l|/h)}\,|_{l=j-k>0} \end{cases} \tag{3.5}
$$

and

$$
x_j = e^{h\gamma\theta_j/2}, \quad j = 1,2; \quad x_{12} = \frac{x_1}{x_2}; \quad q = -e^{-i\pi\gamma}, \quad \gamma = \frac{4\pi}{\beta^2} - 1. \tag{3.6}
$$

The function ρ^{11} is given by the following expression [11]:

$$
\begin{aligned}
\rho^{11}(\theta_{12}) = & \frac{\Gamma(1 + h\gamma i\,\theta_{12}/2\pi)\Gamma(1 - h\gamma i\,\theta_{12}/2\pi - \gamma)}{2\pi i}\,\frac{\sinh(\theta_{12}/2 + i\pi/h)}{\sinh(\theta_{12}/2 - i\pi/h)} \\
& \times \prod_{k=1}^{\infty} \frac{F_k(\theta_{12})\,F_k(2\pi i/h - \theta_{12})}{F_k(2\pi i/h + \theta_{12})\,F_k(2\pi i - \theta_{12})},
\end{aligned} \tag{3.7}
$$

where

$$
F_k(\theta_{12}) = \frac{\Gamma(1 + h\gamma i\,\theta_{12}/2\pi + hk\gamma)}{\Gamma(h\gamma i\,\theta_{12}/2\pi + (hk+1)\gamma)}. \tag{3.8}
$$

In principle, S^{11} is enough to describe the quantum affine Toda field theory since the remaining S-matrices for solitons in the other fundamental representations can be determined by adopting a bootstrap procedure. Most expressions for the remaining S-matrices are neither needed nor provided, though a description of the soliton states A^a_j in the representation a in terms of states A^1_j in the first representation will be given and used in the next section. The scattering matrix S^{12} for a_2 will be used later and is provided in appendix A.

The bootstrap linking states lying in two different representations is given schematically

$$
A^c_i(\theta) \equiv \sum_{j,k} c_i{}^{jk}\,A^a_j(\theta - \theta_p/2)\,A^b_k(\theta + \theta_p/2), \quad l^c_i = l^a_j + l^b_k, \tag{3.9}
$$

where θ_p is the location of the pole in the scattering matrix S^{ab} corresponding to a soliton in the representation labelled c. For instance, starting from the operator A^1_j, for which the scattering S-matrix is known, the solitons in the second representation will be represented by

$$
\begin{aligned}
A^2_i(\theta) &\equiv \sum_{j,k} c_i{}^{jk}\,A^1_j(\theta - i\pi/h)\,A^1_k(\theta + i\pi/h), \qquad l^2_i = l^1_j + l^1_k, \\
c_i{}^{jk} &= (-q)^{-(1-2|j-k|/h)}\,c_i{}^{kj}, \quad j < k = 1, \ldots, h.
\end{aligned} \tag{3.10}
$$

Note, each weight in the second representation can be expressed in only one way as a sum of weights in the first representation. Hence, the sum in (3.10) contains just two terms related as shown. Iterating this process allows a formal presentation for all the states in each fundamental representation.

4. Functional integral approach to the transmission matrix

Before considering in detail all solutions to the triangle equations that express the compatibility between the bulk S-matrix and the transmission matrix - and bearing in mind there are likely to be several formal solutions to the triangle equations, not all of which might be relevant to the present problem - it is worth extending the functional integral argument introduced in [8]. This will supply some constraints that will be helpful in discriminating among the variety of solutions. In particular, the functional integral allows a comparison between the elements of the transmission matrix describing the evolution of field configurations in the presence of a defect labelled by a pair of roots (r, s) and the evolution of field configurations in the presence of the defect labelled by $(0, 0)$. The basic idea is to shift the fields by setting

$$\phi \to \phi - \frac{2\pi r}{\beta}, \quad \psi \to \psi - \frac{2\pi s}{\beta},$$

and use the invariance of the bulk action and the defect potential. The remaining pieces of (2.8), the terms linear in time derivatives, lead to the expression

$$T(r, s) = e^{i\tau(r,s)}\, T(0, 0), \tag{4.1}$$

where

$$\tau(r, s) = \frac{\pi}{\beta}\left(-\delta\phi \cdot (Er + Ds) + (rD + sE) \cdot \delta\psi\right), \tag{4.2}$$

and $\delta\phi,\ \delta\psi$ are the changes in the field configurations from initial to final states.

A soliton passing the defect will either retain its topological charge λ, or its charge will change to μ, one of the other weights within the representation to which the soliton belongs. Thus, the effect of a soliton passing a defect must be to change the defect labels by

$$r \to r - \lambda, \quad s \to s - \mu, \tag{4.3}$$

and, therefore,

$$\delta\phi = -\frac{2\pi\lambda}{\beta}, \quad \delta\psi = -\frac{2\pi\mu}{\beta}.$$

Thus,

$$\tau(r, s) = \frac{2\pi^2}{\beta^2}\left(\lambda \cdot (Er + Ds) - (rD + sE) \cdot \mu\right), \tag{4.4}$$

which is written more conveniently (using $D = 1 - E$) as

$$\tau(r, s) = \frac{2\pi^2}{\beta^2}\left(\frac{1}{2}(\lambda - \mu) \cdot (r + s) - (\lambda - \mu) \cdot E(s - r) + \frac{1}{2}(\lambda + \mu) \cdot (s - r)\right). \tag{4.5}$$

In other words, using this argument it is expected that

$$T(r, s, \lambda, \mu) = Q^{[(\lambda-\mu)\cdot p - 2(\lambda-\mu)\cdot E\alpha + (\lambda+\mu)\cdot\alpha]/4}\, T(0, 0, \lambda, \mu), \quad Q \equiv e^{4\pi^2 i/\beta^2} = q^{-1}, \tag{4.6}$$

where $p = s + r$ and $\alpha = s - r$. Naturally, this style of argument can give no information concerning the rapidity dependence of the transmission matrix but it does suggest, assuming the conservation of topological charge, that a general element of the transmission matrix should have the form:

$$T^{\mu\beta q}_{\lambda\alpha p}(\theta) = Q^{[(\lambda-\mu)\cdot p-2(\lambda-\mu)\cdot E\alpha+(\lambda+\mu)\cdot\alpha]/4}\, T^{\mu}_{\lambda}(\theta)\, \delta^{\beta-\lambda+\mu}_{\alpha}\, \delta^{q+\lambda+\mu}_{p} \tag{4.7}$$

Also, the dependence on p can be eliminated using the unitary transformation

$$U^{\beta s}_{\alpha r} = Q^{\alpha\cdot r/4}\delta^{\beta}_{\alpha}\delta^{s}_{r}, \tag{4.8}$$

to find:

$$T^{\mu\beta q}_{\lambda\alpha p}(\theta) = Q^{[2\alpha\cdot E(\lambda-\mu)+(\alpha+\lambda)^2-(\alpha-\mu)^2]/4}\, T^{\mu}_{\lambda}(\theta)\, \delta^{\beta-\lambda+\mu}_{\alpha}\, \delta^{q+\lambda+\mu}_{p}. \tag{4.9}$$

For the fundamental representations of a_r, labelled $a = 1$ or r, the weights have equal length and the expression simplifies a little. Thus, for solitons in the first representation, equation (4.9) simplifies to

$$T^{1j\beta q}_{\ \ i\alpha p}(\theta) = Q^{\,\alpha\cdot[E(l_i-l_j)+l_i+l_j]/2}\, T^{1\,j}_{\ \ i}(\theta)\, \delta^{\beta-l_i+l_j}_{\alpha}\, \delta^{q+l_i+l_j}_{p}, \tag{4.10}$$

where the general weights μ, λ have been replaced by the weights lying in the first representation, namely $l^1_i \equiv l_i$ (see (3.3)). Further, when these specific weights are used as labels, the notation $T^{1\,j}_{\ \ i}$ is used as a simplification. A similar expression holds for the transmission matrix for solitons in the representation r; for these, the topological charges are merely a sign change relative to those in the first representation ($l^r_k \equiv l_{\bar{k}} = -l^1_k$). For the case of a_1, or the sine-Gordon model, $E = 0$ and (4.10) agrees with the findings of [8].

Before proceeding to solve the triangle equations for the a_2 model, it is instructive to apply the bootstrap procedure to the general form of T^1, given in (4.10), to see the extent to which it is possible, solely from the bootstrap, to gather information about the classical quantity E and the still undetermined part of the transmission matrix. For this purpose, consider D_α to be the defect operator. Then, it is formally possible to describe the interaction between a defect and a soliton within the first fundamental representation as follows ($\theta > 0$),

$$A^1_i(\theta)D_\alpha = T^{1j\beta}_{\ \ i\alpha}(\theta)D_\beta A^1_j(\theta). \tag{4.11}$$

Note, the indices p and q do not appear in (4.11) since, as already established, the transmission matrix does not depend on them. Note also that (4.11) is consistent with the notation (3.4) used for the S-matrix. The interaction between the defect and solitons in the second representation will be represented by

$$\begin{aligned} A^2_i(\theta)\, D_\alpha &= T^{2m\beta}_{\ \ i\alpha}(\theta)D_\beta A^2_m(\theta) \\ &= c_i^{\ jk}\, T^{1a\delta}_{\ \ j\beta}(\theta - i\pi/h)\, T^{1b\beta}_{\ \ k\alpha}(\theta + i\pi/h)\ D_\delta\, A^1_a(\theta - i\pi/h)\, A^1_b(\theta + i\pi/h) \\ &= T^{2n\delta}_{\ \ i\alpha}(\theta)\, c_n^{\ ab}\, D_\delta A^1_a(\theta - i\pi/h)\, A^1_b(\theta + i\pi/h) \end{aligned} \tag{4.12}$$

where all repeated indices are summed. Thus,

$$T^{2n\delta}_{\ \ i\alpha}(\theta)\, c_n^{\ ab} = c_i^{\ jk}\, T^{1a\delta}_{\ \ j\beta}(\theta - i\pi/h)\, T^{1b\beta}_{\ \ k\alpha}(\theta + i\pi/h). \tag{4.13}$$

Bearing in mind the result obtained for the transmission matrix in the sine-Gordon model [8], and also noting the rapidity dependence within the S-matrix (3.5), a suitable ansatz to adopt for the rapidity independent part of (4.10) is

$$T^{1j}_{\ i}(\theta) = t_{ij}\, x^{\epsilon_{ij}}\, g^1(\theta), \tag{4.14}$$

where t_{ij} and ϵ_{ij} are constants, and $g(\theta)$ is independent of the soliton labels.

When $a = b$, the right hand side of (4.13) must vanish since there are no weights of the form $2l^1_a$ in the second representation. As a consequence, the following relations must hold

$$Q^{\,l_k\cdot El_j + l_a\cdot(1+E)(l_k-l_j)}\, c_i^{\ jk} = -(-)^{\epsilon_{ka}-\epsilon_{ja}} Q^{\epsilon_{ja}-\epsilon_{ka}}\, c_i^{\ kj}. \tag{4.15}$$

Putting j equal to the index a, and using the fact (deducible from (3.3)) that $l_a\cdot(l_a - l_k) = 1$ for all $k \neq a$, (4.15) becomes

$$c_i^{\ ak} = -(-)^{\epsilon_{ka}-\epsilon_{aa}} Q^{1+\epsilon_{aa}-\epsilon_{ka}}\, c_i^{\ ka}, \tag{4.16}$$

and this can be compared with (3.10). Firstly, interchanging a and k requires

$$\epsilon_{ak} + \epsilon_{ka} - \epsilon_{kk} - \epsilon_{aa} = 2, \quad a \neq k. \tag{4.17}$$

Secondly, using (3.10) leads to more detailed information, namely,

$$\epsilon_{ka} = \epsilon_{aa} + 2|a-k|/h, \quad a < k\,. \tag{4.18}$$

Together, (4.17) and (4.18) determine all the off-diagonal exponents appearing in the transmission matrix in terms of its diagonal exponents. Also, when $j < k \neq a$ it is possible to gather information concerning the matrix E because (4.15) demands

$$(l_k - l_a)\cdot E(l_j - l_a) = -1,\ j<k<a \text{ or } a<j<k\,; \quad (l_k - l_a)\cdot E(l_j - l_a) = 1,\ j<a<k. \tag{4.19}$$

Because of equivalences it is sufficient to consider only one of these sets of relations. Making use of (3.3), the constraints implied by (4.19) can be rewritten

$$(\alpha_k + \alpha_{k+1} + \cdots + \alpha_{a-1})\cdot E(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{k-1}) = -1,\ j<k<a. \tag{4.20}$$

The independent relations provided by (4.20) state the following

$$\alpha_l\cdot E\alpha_m = \begin{cases} -1 & m = l-1, \\ \ \ 0 & m = 1,\dots l-2, \end{cases} \qquad l = 2,\dots,a-1. \tag{4.21}$$

Since the index a takes the values$= 1,\dots,h$, the total number of independent constraints in (4.21) is $r(r-1)/2$ and precisely equal to the number of degrees of freedom of the matrix E. Consequently, E is completely determined by the bootstrap procedure and can be compared with the formula (2.11) which defined the matrix $E = 1 - D$ established in the classical setting of the defect problem. It can be seen that the two expressions coincide, provided the clockwise cyclic permutation of the simple roots in formula (2.11) is chosen.

Next, consider the terms for which $a \neq b$ and $l_i^2 = l_a^1 + l_b^1$ in (4.13). Then, $n = i$ and the left hand side of (4.13) can be written in two ways according to the choice of ordering a with respect to b. Thus,

$$\begin{aligned} T^{2}{}^{i\gamma}_{i\alpha} &= T^{1}{}^{a\gamma}_{a\beta}(\theta - i\pi/h)T^{1}{}^{b\beta}_{b\alpha}(\theta + i\pi/h) + \left(\frac{c_i^{ba}}{c_i^{ab}}\right) T^{1}{}^{a\gamma}_{b\beta}(\theta - i\pi/h)T^{1}{}^{b\beta}_{a\alpha}(\theta + i\pi/h) \\ &= T^{1}{}^{b\gamma}_{b\beta}(\theta - i\pi/h)T^{1}{}^{a\beta}_{a\alpha}(\theta + i\pi/h) + \left(\frac{c_i^{ab}}{c_i^{ba}}\right) T^{1}{}^{b\gamma}_{a\beta}(\theta - i\pi/h)T^{1}{}^{a\beta}_{b\alpha}(\theta + i\pi/h). \end{aligned} \quad (4.22)$$

Since the right hand sides must match, and since the dependence on θ is different in different terms, there is an additional pair of constraints. Specifically, these are

$$\epsilon_{aa} = \epsilon_{bb} \equiv \epsilon; \quad \epsilon_{ab} - \epsilon_{ba} = 2(1 - 2|a - b|/h), \quad a < b = 1, \dots, h. \quad (4.23)$$

Therefore, all the diagonal exponents in the transmission matrix are the same; the relations in the second group are precisely the differences of the relations given earlier in (4.18). Using (4.18) and (4.23) the diagonal terms of the transmission matrix for the solitons in the second representation are

$$T^{2}{}^{i\gamma}_{i\alpha}(\theta) = Q^{\alpha\cdot(l_a+l_b)}\, x^{2\epsilon}(t_{aa}t_{bb} + x^2 t_{ab}t_{ba})\, g^1(\theta - i\pi/h)\, g^1(\theta + i\pi/h)\, \delta_\alpha^\gamma, \quad l_i^2 = l_a + l_b. \quad (4.24)$$

It is possible to go a little further in the analysis of (4.13) by looking at those cases for which

$$\frac{c_i{}^{jk}}{c_i{}^{kj}} = \frac{c_n{}^{ab}}{c_n{}^{ba}}. \quad (4.25)$$

Since l_n^2 is uniquely $l_a^1 + l_b^1$ there is only the choice of ordering a and b when considering (4.13). Because of this, and the fact $c_n{}^{ab} \neq c_n{}^{ba}$, there must be further constraints on $T^1(\theta)$. With the particular choice (4.25), these are

$$\begin{aligned} &T^{1}{}^{a\delta}_{j\beta}(\theta - i\pi/h)\, T^{1}{}^{b\beta}_{k\alpha}(\theta + i\pi/h) + \frac{c_i{}^{kj}}{c_i{}^{jk}}\, T^{1}{}^{a\delta}_{k\beta}(\theta - i\pi/h)\, T^{1}{}^{b\beta}_{j\alpha}(\theta + i\pi/h) \\ &\qquad = \frac{c_i{}^{jk}}{c_i{}^{kj}}\, T^{1}{}^{b\delta}_{j\beta}(\theta - i\pi/h)\, T^{1}{}^{a\beta}_{k\alpha}(\theta + i\pi/h) + T^{1}{}^{b\delta}_{k\beta}(\theta - i\pi/h)\, T^{1}{}^{a\beta}_{j\alpha}(\theta + i\pi/h). \end{aligned} \quad (4.26)$$

For definiteness, suppose that $j < k$. Then, the constraint (4.25) is satisfied (using (3.10)) provided $|a - b| = |j - k|$ (if $a < b$), or $|a - b| = h - |j - k|$ (if $a > b$). The full set of possibilities will not be analysed here and to illustrate some important points only the two simplest cases, namely $|j - k| = 1$ and $|j - k| = h - 1$, will be considered in detail. Besides, these cover all the possibilities for a_2. Bearing in mind that

$$l_i^2 = l_j + l_k, \quad l_n^2 = l_a + l_b, \quad (4.27)$$

it is useful to list explicitly the combinations of indices j, k, a, b which will be investigated. Firstly, consider $|j - k| = 1$. For the a_2 case such combinations are

$$\begin{aligned} &l_i^2 = l_1 + l_2,\ l_n^2 = l_2 + l_3 \ \text{ if } a < b; \quad l_i^2 = l_2 + l_3,\ l_n^2 = l_3 + l_1 \ \text{ if } a > b \\ &l_i^2 = l_2 + l_3,\ l_n^2 = l_1 + l_2 \ \text{ if } a < b; \quad l_i^2 = l_1 + l_2,\ l_n^2 = l_3 + l_1 \ \text{ if } a > b, \end{aligned} \quad (4.28)$$

while their generalizations for the a_r affine Toda field theory are

$$l_i^2 = l_j + l_{j+1},\; l_n^2 = l_{j+1} + l_{j+2} \;\text{ if } a < b;\; l_i^2 = l_{h-1} + l_h,\; l_n^2 = l_h + l_1 \;\text{ if } a > b \quad (4.29)$$

$$l_i^2 = l_j + l_{j+1},\; l_n^2 = l_{j-1} + l_j \;\text{ if } a < b;\; l_i^2 = l_1 + l_2,\; l_n^2 = l_h + l_1 \;\text{ if } a > b. \quad (4.30)$$

It should be emphasised that while (4.28) represents all possible combinations of indices for a_2 with the constraint $|j-k| = 1$, (4.29) (4.30) only provides a subset of the possibilities for a_r. Using (4.17), (4.18), (4.19), (4.23), it can be verified that equation (4.26) is an identity for the weights in (4.29). The corresponding transmission matrix elements for solitons in the second representation are

$$\begin{aligned} T^{2n\delta}_{i\alpha}(\theta) = {} & Q^{\alpha\cdot[E(l_j-l_{j+2})+2l_{j+1}+l_j+l_{j+2}]/2}\, x^{2(\epsilon+1-2/h)}(t_{jj+2}t_{j+1j+1} + x^2 t_{j+1j+2}t_{jj+1}) \\ & \times g^1(\theta - i\pi/h)\, g^1(\theta + i\pi/h)\, \delta_\alpha^{\delta - l_j + l_{j+2}}, \end{aligned} \quad (4.31)$$

with

$$l_i^2 = l_j + l_{j+1}, \qquad l_n^2 = l_{j+1} + l_{j+2} \qquad j = 1, \ldots h-2$$

and

$$\begin{aligned} T^{2n\delta}_{i\alpha}(\theta) = {} & Q^{\alpha\cdot[E(l_{h-1}-l_1)+2l_h+l_{h-1}+l_1]/2}\, x^{2(\epsilon+1-2/h)}(t_{h-11}t_{hh} + x^2 t_{h1}t_{h-1h}) \\ & \times g^1(\theta - i\pi/h)\, g^1(\theta + i\pi/h)\, \delta_\alpha^{\delta - l_{h-1} + l_1}, \end{aligned} \quad (4.32)$$

with

$$l_i^2 = l_{h-1} + l_h, \qquad l_n^2 = l_h + l_1.$$

Alternatively, using the index combinations in (4.30), the expression (4.26) is satisfied provided the following constraints on the constants t_{ij} hold

$$t_{jj}t_{j+1j-1} = t_{j+1j}t_{jj-1} \qquad j = 2, \ldots, h-1; \qquad t_{11}t_{2h} = t_{21}t_{1h}, \quad (4.33)$$

and the corresponding elements of the transmission matrix for the solitons in the second representations are equal to zero.

Finally, the case $|i-j| = h-1$ corresponds to the following two-index combinations for a_2 (the possibility $a < b$ having been investigated already):

$$\begin{aligned} & l_i^2 = l_1 + l_3, \quad l_n^2 = l_2 + l_1 \quad \text{if} \quad a > b, \\ & l_i^2 = l_1 + l_3, \quad l_n^2 = l_3 + l_2 \quad \text{if} \quad a > b; \end{aligned} \quad (4.34)$$

and these generalize for the a_r affine Toda field theory to

$$l_i^2 = l_1 + l_h, \quad l_n^2 = l_2 + l_1 \qquad \text{if} \quad a > b, \quad (4.35)$$

$$l_i^2 = l_1 + l_h, \quad l_n^2 = l_h + l_{h-1} \quad \text{if} \quad a > b. \quad (4.36)$$

Just as in the previous case, using the weights in (4.35), the expression (4.26) is an identity with the corresponding transmission matrix element given by

$$\begin{aligned} T^2(\theta)^{n\,\beta}_{i\,\alpha} = {} & Q^{\alpha\cdot[E(l_h-l_1)+2l_1+l_2+l_h]/2} x^{2(\epsilon+1-2/h)} \\ & \times (t_{11}t_{h2} + x^2 t_{h1}t_{12})\, g^1(\theta - i\pi/h)\, g^1(\theta + i\pi/h)\, \delta_\alpha^{\beta - l_h + l_2}, \end{aligned} \quad (4.37)$$

with

$$l_i^2 = l_1 + l_h, \qquad l_n^2 = l_2 + l_1.$$

On the other hand, using the index combination in (4.36), the expression (4.26) forces the following constraint on the constants t_{ij},

$$t_{hh} t_{1h-1} = t_{1h} t_{hh-1}, \tag{4.38}$$

with the corresponding transmission matrix elements being equal to zero.

In summary, this partial analysis of the bootstrap procedure determines the matrix T^1 for all the affine Toda field theories up to a function $g(\theta)$ that is independent of the soliton labels, and up to constants t_{ij}, which are themselves constrained. Moreover, it has been noted that provided the initial T^1 matrix has all entries different from zero, the transmission matrix T^2 is required to have at least some off-diagonal entries equal to zero. For the simplest case of a_2, the analysis based on the bootstrap has been carried out completely, and therefore it is possible to write down the full T^2 matrix for antisolitons.

To conclude, the transmission matrices for the a_2 affine Toda field theory predicted by the bootstrap procedure are

$$T^1{}_{i\alpha}^{n\beta}(\theta) = g^1(\theta) \begin{pmatrix} t_{11}\, Q^{\alpha\cdot l_1}\, \delta_\alpha^\beta & t_{12}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_1} & t_{13}\, x^{2/3}\, Q^{-\alpha\cdot l_2}\, \delta_\alpha^{\beta+\alpha_0} \\ t_{21}\, x^{2/3}\, Q^{-\alpha\cdot l_3}\, \delta_\alpha^{\beta+\alpha_1} & t_{22}\, Q^{\alpha\cdot l_2}\, \delta_\alpha^\beta & t_{23}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_2} \\ t_{31}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_0} & t_{32}\, x^{2/3}\, Q^{-\alpha\cdot l_1}\, \delta_\alpha^{\beta+\alpha_2} & t_{33}\, Q^{\alpha\cdot l_3}\, \delta_\alpha^\beta \end{pmatrix} \tag{4.39}$$

$$\begin{aligned} T^2{}_{\bar{i}\alpha}^{\bar{n}\beta}(\theta) &= g^1(\theta - i\pi/3)\, g^1(\theta + i\pi/3) \left(1 + x^2 \frac{t_{21}t_{31}t_{13}}{t_{11}t_{22}t_{33}}\right) \\ &\times \begin{pmatrix} t_{11}\, Q^{-\alpha\cdot l_1}\, \delta_\alpha^\beta & t_{21}t_{33}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_1} & 0 \\ 0 & t_{22}\, Q^{-\alpha\cdot l_2}\, \delta_\alpha^\beta & t_{32}t_{11}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_2} \\ t_{13}t_{22}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_0} & 0 & t_{33}\, Q^{-\alpha\cdot l_3}\, \delta_\alpha^\beta \end{pmatrix}, \end{aligned} \tag{4.40}$$

with

$$\frac{t_{21}\, t_{13}}{t_{11}} = t_{23}, \qquad \frac{t_{32}\, t_{21}}{t_{22}} = t_{31}, \qquad \frac{t_{32}\, t_{13}}{t_{33}} = t_{12}. \tag{4.41}$$

Note, the function $g^1(\theta)$ has been redefined in order to absorb the factor x^ϵ.

At first sight the imbalance between solitons and antisolitons appears strange and one might wonder about its consistency since the bootstrap could be run the other way to define T^1 starting with T^2. Although the details will not be given here the results are entirely consistent; starting with a matrix containing these zeroes and using it to define T^1 does indeed recover (4.39).

In the next section it will be shown that the transmission matrix (4.39) coincides with a solution of the triangle equations.

5. The transmission matrix for the a_2 affine Toda field theory: the triangle equations

In this section arguments will be restricted to the special case a_2. In order to find the transmission matrices describing the interaction between the jump-defect and solitons, the general procedure applied successfully for the sine-Gordon model in [8] will be adopted. The first step is to solve the triangle equations, which relate the elements of the transmission matrix to S-matrix elements; and, in the first instance, attention will be focused on the transmission matrix for solitons in the $a=1$ representation.[1] Consequently, the triangle equation reads

$$S^{11}{}^{mn}_{kl}(\theta_{12})\, T^{1}{}^{t\beta}_{n\alpha}(\theta_1)\, T^{1}{}^{s\gamma}_{m\beta}(\theta_2) = T^{1}{}^{n\beta}_{l\alpha}(\theta_2)\, T^{1}{}^{m\gamma}_{k\beta}(\theta_1)\, S^{11}{}^{st}_{mn}(\theta_{12}), \tag{5.1}$$

where the elements of the transmission matrix T^1 are infinite dimensional. As noted in the previous section, the transmission matrix elements have two types of label. The roman labels stand for the soliton states $1,2,3$, while the greek labels represent vectors in the weight lattice. Because of the topological charge conservation, the elements of the transmission matrix can be written as follows

$$T^{1}{}^{n\beta}_{i\alpha}(\theta) = t^{1}{}^{n}_{i\alpha}(\theta)\; \delta_{\alpha}^{\beta-l_i+l_n}, \quad i,n=1,2,3 \tag{5.2}$$

where l_i, l_n are the weights (3.3), which in the case of a_2 are

$$l_1 = \frac{1}{3}(2\alpha_1+\alpha_2), \quad l_2 = -\frac{1}{3}(\alpha_1-\alpha_2), \quad l_3 = -\frac{1}{3}(\alpha_1+2\alpha_2). \tag{5.3}$$

In the following discussion, indices referring to the first representation will be omitted since there is no possibility of confusion; for instance, the matrix T^1 will be indicated simply by T, and so on. Using the ansatz (5.2) for the transmission matrix and the S-matrix (3.5), it is possible to find solutions to the triangle equation (5.1) up to an overall scalar function of the rapidity. A classification of all possible solutions, and a detailed explanation of the procedure adopted to obtain them, is available in appendix B. Among all the solutions listed (B.23) coincides with the T matrix (4.39) discovered already by analysing the bootstrap procedure, as explained in the previous section. Notice that apart from an overall scale this solution contains eight parameters t_{ij}, satisfying the three relations (4.41). However, using suitably designed unitary transformations most of this freedom can be removed to leave just one essential parameter. To demonstrate this a slightly more general element of the transmission matrix T will be considered instead of the expression (5.2), namely

$$T^{n\beta p}_{i\alpha q}(\theta) = t^{n}_{i\alpha p}(\theta)\; \delta_{\alpha}^{\beta-l_i+l_n}\; \delta_{p}^{q+l_i+l_n}. \tag{5.4}$$

This was the general expression considered using the functional integral approach and the extra delta function does not alter the solutions to the triangle equation (5.2).

It is convenient to split the argument into two steps. Consider the solution (B.23) and multiply it by $(t_{11}t_{22}t_{33})^{-1/3}$. Next, conjugate the matrix using the unitary transformation:

$$W^{\beta q}_{\alpha p} = (t_{11}^{-p\cdot l_1/2}\, t_{22}^{-p\cdot l_2/2}\, t_{33}^{-p\cdot l_3/2})\; \delta^{\beta}_{\alpha}\delta^{q}_{p}, \qquad |t_{11}|=|t_{22}|=|t_{33}|=1. \tag{5.5}$$

[1] From now on, these solitons will be called simply solitons, while the solitons in the $a=2$ representation will be called antisolitons.

After conjugation, the parametric part of the solution (B.23) is modified and represented schematically as follows,

$$\begin{pmatrix} 1 & t_{12}(t_{11}t_{22})^{-1/2} & t_{13}(t_{11}t_{33})^{-1/2} \\ t_{21}(t_{11}t_{22})^{-1/2} & 1 & t_{23}(t_{22}t_{33})^{-1/2} \\ t_{31}(t_{11}t_{33})^{-1/2} & t_{32}(t_{22}t_{33})^{-1/2} & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & \hat{t}_{12} & \hat{t}_{13} \\ \hat{t}_{21} & 1 & \hat{t}_{23} \\ \hat{t}_{31} & \hat{t}_{32} & 1 \end{pmatrix}. \tag{5.6}$$

Next, conjugate using the unitary transformation,

$$V_{\alpha p}^{\beta q} = (\hat{t}_{21}^{-\alpha\cdot\alpha_1/3}\hat{t}_{13}^{-\alpha\cdot\alpha_0/3}\hat{t}_{32}^{-\alpha\cdot\alpha_2/3})\,\delta_\alpha^\beta\delta_p^q, \qquad |\hat{t}_{21}| = |\hat{t}_{13}| = |\hat{t}_{32}| = 1, \tag{5.7}$$

which, together with (4.41), transforms (5.6) to a matrix depending on a single parameter t, which can be represented schematically by,

$$\begin{pmatrix} 1 & t^{2/3} & t^{1/3} \\ t^{1/3} & 1 & t^{2/3} \\ t^{2/3} & t^{1/3} & 1 \end{pmatrix}, \tag{5.8}$$

with $t \equiv (\hat{t}_{21}\hat{t}_{13}\hat{t}_{32}) = (t_{21}t_{13}t_{32})/(t_{11}t_{22}t_{33})$. Consequently, solutions (4.39) and (4.40) become, respectively

$$T^{1n\beta}_{\ i\alpha}(\theta) = g^1(\theta)\begin{pmatrix} Q^{\alpha\cdot l_1}\,\delta_\alpha^\beta & \hat{x}^2\,\delta_\alpha^{\beta-\alpha_1} & \hat{x}\,Q^{-\alpha\cdot l_2}\,\delta_\alpha^{\beta+\alpha_0} \\ \hat{x}\,Q^{-\alpha\cdot l_3}\,\delta_\alpha^{\beta+\alpha_1} & Q^{\alpha\cdot l_2}\,\delta_\alpha^\beta & \hat{x}^2\,\delta_\alpha^{\beta-\alpha_2} \\ \hat{x}^2\,\delta_\alpha^{\beta-\alpha_0} & \hat{x}\,Q^{-\alpha\cdot l_1}\,\delta_\alpha^{\beta+\alpha_2} & Q^{\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix}, \tag{5.9}$$

$$T^{2n\beta}_{\ i\alpha}(\theta) = g^2(\theta)\begin{pmatrix} Q^{-\alpha\cdot l_1}\,\delta_\alpha^\beta & \hat{x}\,\delta_\alpha^{\beta+\alpha_1} & 0 \\ 0 & Q^{-\alpha\cdot l_2}\,\delta_\alpha^\beta & \hat{x}\,\delta_\alpha^{\beta+\alpha_2} \\ \hat{x}\,\delta_\alpha^{\beta+\alpha_0} & 0 & Q^{-\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix}, \tag{5.10}$$

where it has been convenient to set

$$t \equiv e^{-3\gamma\Delta},$$

and

$$g^2(\theta) = g^1(\theta - i\pi/3)\; g^1(\theta + i\pi/3)\;(1+\hat{x}^3), \quad \hat{x} = e^{\gamma(\theta-\Delta)}. \tag{5.11}$$

Eventually, the constant Δ will be related to the Lagrangian parameter $\sigma = e^{-\eta}$ introduced in (2.17).

In the next section these solutions to the triangle equations will be used as suitable candidates for describing the jump-defect problem and an additional constraint will be introduced to determine the scalar function g up to a CDD factor. Though the subsequent analysis will rely on solution (5.9), it could be anticipated that this is not the only relevant solution of the triangle equation. Evidence that it is the appropriate solution will be provided, as well as reasons why the functional integral approach selects solution (B.23) among all the solutions presented in appendix B.

6. The transmission matrix for the a_2 model: additional constraints

Additional constraints are necessary to determine the overall factor $g(\theta)$ in the solutions (5.9) and (5.10). For unitary theories these constraints are based on unitarity and crossing properties of the S-matrix, although it was found convenient in [8] to use equivalent constraints based on unitarity and 'annihilation poles'. The latter was found to be more suitable when analysing the sine-Gordon system because it avoided having to relate scattering of solitons arriving at the defect from the left to the scattering of solitons arriving at the defect from the right. In the present context, the theory is not unitary, the S-matrix is not a unitary matrix for real rapidity, and it is not expected that the transmission matrix should be unitary. For this reason, the methods used previously to analyse the sine-Gordon model will need to be adjusted slightly.

However, although the S-matrix is not unitary it is nevertheless natural to assume that

$$S(-\theta) = [S(\theta)]^{-1} \tag{6.1}$$

and therefore, a similar relation is also supposed to hold for the transmission matrix [9]. The condition is

$$T^{1}{}_{a\alpha}^{b\beta}(\theta)\, \tilde{T}^{1}{}_{b\beta}^{c\gamma}(-\theta) = \delta_a^c \delta_\alpha^\gamma, \tag{6.2}$$

where $\tilde{T}^1$ is the transmission matrix describing the interaction between the defect and a soliton travelling from the right to the left. In fact, since parity is violated explicitly in the jump-defect problem, the matrix $\tilde{T}^1$ is expected to differ from the matrix T^1 that describes solitons travelling from left to right. Indeed, the triangle equation satisfied by $\tilde{T}^1$ is

$$S^{11}{}_{lk}^{nm}(\theta_{12})\, \tilde{T}^{1}{}_{n\alpha}^{t\beta}(\theta_1)\, \tilde{T}^{1}{}_{m\beta}^{s\gamma}(\theta_2) = \tilde{T}^{1}{}_{l\alpha}^{n\beta}(\theta_2)\, \tilde{T}^{1}{}_{k\beta}^{m\gamma}(\theta_1)\, S^{11}{}_{nm}^{ts}(\theta_{12}); \tag{6.3}$$

and this differs slightly from the relation (5.1) previously discussed. Consequently, the solutions of these two triangle equations are not the same. Nevertheless, $\tilde{T}^1(-\theta)$ is the inverse of $T^1(\theta)$ and, therefore,

$$\tilde{T}^{1}{}_{i\alpha}^{n\beta}(-\theta) = \frac{1}{g^1(\theta)}\,\frac{1}{1-Q\hat{x}^3}\begin{pmatrix} Q^{-\alpha\cdot l_1}\,\delta_\alpha^\beta & 0 & -Q\hat{x}\,\delta_\alpha^{\beta+\alpha_0} \\ -Q\hat{x}\,\delta_\alpha^{\beta+\alpha_1} & Q^{-\alpha\cdot l_2}\,\delta_\alpha^\beta & 0 \\ 0 & -Q\hat{x}\,\delta_\alpha^{\beta+\alpha_2} & Q^{-\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix}. \tag{6.4}$$

It is worth pointing out that requiring T^1 to have an inverse is already a constraint, since not all solutions to the triangle equations will have this property.

Crossing requires that (6.4) should be closely related to the transmission matrix T^2 for the antisoliton via the relation

$$T^{2}{}_{n\alpha}^{i\beta}(\theta) = \tilde{T}^{1}{}_{i\alpha}^{n\beta}(i\pi-\theta). \tag{6.5}$$

Comparing (6.4) with (5.10) it is clear (6.5) will be satisfied provided

$$g^1(\theta)\, g^1(\theta+i2\pi/3)\, g^1(\theta+i4\pi/3)\, \left(1-Q^3\hat{x}^3\right)\,\left(1-Q\hat{x}^3\right) = 1, \tag{6.6}$$

which clearly constrains the overall factor $g^1(\theta)$. It is interesting that an expression for T^2 emerges containing the zeroes remarked upon before, previously generated by the bootstrap.

A minimal solution to (6.6) is provided by setting

$$g^1(\theta) = \frac{f(\theta)}{(2\pi)^{2/3}\,\hat{x}} \tag{6.7}$$

with

$$f(\theta) = \Gamma[(1+\gamma)/2 - z] \prod_{k=1}^{\infty} \frac{\Gamma[(1+\gamma)/2 + 3k\gamma - z]\,\Gamma[(1-\gamma)/2 + (3k-2)\gamma + z]}{\Gamma[(1-\gamma)/2 + 3k\gamma + z]\,\Gamma[(1+\gamma)/2 + (3k-1)\gamma - z]}\,, \tag{6.8}$$

where

$$z = \frac{i3\gamma(\theta - \Delta)}{2\pi}.$$

Using (5.11) a matching expression can be found for $g^2(\theta)$:

$$g^2(\theta) = \frac{\Gamma[1/2 + \gamma - z]}{(2\pi)^{1/3}\,\hat{x}^{1/2}} \prod_{k=1}^{\infty} \frac{\Gamma[1/2 + (3k+1)\gamma - z]\,\Gamma[(1/2 + (3k-2)\gamma + z]}{\Gamma[(1/2 + (3k-1)\gamma + z]\,\Gamma[(1/2 + (3k-1)\gamma - z]}\,. \tag{6.9}$$

However, these expressions could be modified by multiplying $g^1(\theta)$ by any function $h(\theta)$ that satisfies

$$h(\theta)\,h(\theta + i2\pi/3)\,h(\theta + i4\pi/3) = 1,$$

and the ambiguity is not resolvable without comparing the results of the algebraic manipulations with the outcome of some alternative dynamical calculations. Unfortunately, such calculations are beyond the scope of this article.

Since crossing has been used to constrain $g^1(\theta)$, and since the theory is not unitary, there should be no further constraints. Previously, in [8], it was found convenient to use the unitarity of the sine-Gordon model alongside the annihilation poles. However, examining the annihilation poles in the present context merely reproduces (6.6). The 'annihilation pole' condition is provided by a virtual process where a particle and its antiparticle annihilate to the vacuum and is described schematically by the following expression

$$c_0{}^{\bar{a}a}\delta_\alpha^\beta = \sum_e T^2{}^{\bar{a}\beta}_{\bar{e}\gamma}(\theta - i\pi/2)\,T^1{}^{a\gamma}_{e\alpha}(\theta + i\pi/2)\,c_0{}^{\bar{e}e}. \tag{6.10}$$

To perform the calculation it is necessary to determine the ratios of the couplings appearing in (6.10) by examining the S^{12} and S^{21} matrix elements provided in appendix A. When $\theta_{12} = i\pi$ the couplings are

$$\begin{aligned} c_{i\bar{\imath}}{}^0\,c_0{}^{\bar{k}k} &= c_{\bar{\imath}i}{}^0\,c_0{}^{k\bar{k}} = \rho_0^{12}\,(q - q^{-1}), \quad && i,k = 1,2,3, \\ c_{j\bar{k}}{}^0\,c_0{}^{\bar{k}j} &= c_{\bar{\jmath}k}{}^0\,c_0{}^{k\bar{\jmath}} = 0, && j,k = 1,2,3, \quad j \neq k, \end{aligned} \tag{6.11}$$

where ρ_0^{12} is the scalar function ρ^{12} calculated when $\theta_{12} = i\pi$. As a consequence, the coupling ratios appearing in (6.10) are all equal to one. Then, using the transmission

JHEP07(2007)001

factors (5.9), (5.10) in equation (6.10), and setting $a = 1, 2$ or 3 , the ‘annihilation pole’ condition recovers (6.6).

Consider the pole occurring in the expression for $g^1(\theta)$ at $z = (1 + \gamma)/2$, or in terms of rapidity at

$$\theta_P = \Delta - \frac{i\pi}{3} - \frac{i\pi}{3\gamma}.$$

It is tempting to associate this pole with the complex rapidity at which the classical delay diverges, namely (2.23), especially given that $1/\gamma \to 0$ in the classical limit. That would then require the identification

$$\Delta = \eta + \frac{i\pi}{2}, \tag{6.12}$$

at least in the limit $\beta \to 0$. With this identification, the complex energy of the state associated with the pole at $\theta = \theta_P$ is given by

$$E = m_s \cosh\theta_P = m_s \cosh\eta\, \sin\left(\frac{\pi}{3} + \frac{\pi}{3\gamma}\right) + i m_s \sinh\eta\, \cos\left(\frac{\pi}{3} + \frac{\pi}{3\gamma}\right), \tag{6.13}$$

and this enjoys a positive real part and negative imaginary part provided

$$\frac{1}{2} < \gamma < 2,$$

or, in terms of the coupling, $8\pi/3 > \beta^2 > 4\pi/3$. Thus, it seems this pole appears to indicate an unstable state in the quantum theory that is completely disconnected from any phenomenon in the classical model. This kind of feature did not appear in the analysis of the sine-Gordon model.

There are other reasons for making the identification (6.12), and reasons related to breathers will be discussed in the next section. However, comparison with the sine-Gordon case already provides some additional motivation for the choice. In fact, aligning with the notation used in the present article, the transmission matrix for the sine-Gordon model found in [8] takes the form

$$T^{\mathrm{SG}}{}^{n\beta}_{i\alpha}(\theta) = g^{\mathrm{SG}}(\theta) \begin{pmatrix} Q^{\alpha/2}\,\delta_\alpha^\beta & (-q)^{1/2}\,e^{\gamma(\theta-\eta)}\,\delta_\alpha^{\beta-2} \\ (-q)^{1/2}\,e^{\gamma(\theta-\eta}\,\delta_\alpha^{\beta+2} & Q^{-\alpha/2}\,\delta_\alpha^\beta \end{pmatrix},$$

while for a_2, the transmission matrices found for solitons and antisolitons, namely (5.9) and (5.10), with the choice (6.12) have remarkably similar elements since

$$\hat{x} = e^{\gamma(\theta-\Delta)} = (-q)^{-1/2} e^{\gamma(\theta-\eta)}. \tag{6.14}$$

Of course nothing can be said concerning the manner in which the classical defect parameter η might be renormalised, and in fact notation has been abused slightly (though without leading to any misunderstandings) by using the same symbol in two different contexts.

At this stage, it is possible to compare the a_2 transmission matrices with the available classical results, namely the delays (2.20) experienced by solitons or antisolitons travelling past the defect. Classically, there is little difference in behaviour between the soliton and the antisoliton. In either case the the defect causes a phase shift varying between 0 and

$-2\pi/3$ for the soliton (a=1), and between 0 and $2\pi/3$ for an antisoliton (a=2). Because of this shift, the topological charge (2.4) of a soliton or antisoliton might change as it passes the defect. It was pointed out that the topological charge of a soliton or antisoliton passing through the defect could be converted to just one of the adjacent topological charges. In particular, assuming $\theta > \eta$, the argument of the delay (2.22) will be negative for the soliton, therefore its topological charge l_i will change, if it changes at all, into l_{i+1} (with $(i+1)$ understood modulo 3), while for the antisoliton the argument of the delay will be positive and the topological charge $l_{\bar{i}}$ will change, if it changes, into $l_{\overline{i+1}}$ (with $(i+1)$ understood again modulo 3). Inspecting (5.10), it can be seen that the transmission matrix representing the behaviour of antisolitons provides a good match to the classical situation because of the presence of zeros in expected positions. On the other hand, the transmission matrix for solitons does not possess the expected zeros corresponding to the classical selection rule. It appears that in the quantum context a soliton passing through the defect may change into either of the solitons adjacent to it; although the classically allowed transition remains the most probable, the soliton can tunnel to its classically forbidden neighbour. From this perspective, the defect can act as a filter, which is intriguingly asymmetrical between solitons and antisolitons. This kind of effect was not evident in the sine-Gordon model since there the soliton and antisoliton belong to the same representation when regarded from the perspective offered by the present context, and transitions between the two are never forbidden, either classically or quantum mechanically.

7. Transmission factors for the lightest breathers

In order to collect additional evidence to support the idea that the transmission matrix (5.9) describes a_2 solitons interacting with a jump-defect, the transmission factors for the lightest breathers will be calculated. Since the lightest breathers correspond to the quantum Toda particles described by the fundamental bulk fields appearing in the Lagrangian density, their transmission factors can be compared perturbatively with classical transmission coefficients obtained by linearising the defect conditions (2.14).

The breathers describe scalar bound states whose existence is revealed by the following poles located in the forward channel of the soliton-antisoliton scattering matrices (see appendix A)

$$\theta_k = i\pi\left(1 - \frac{2\,k}{3\gamma}\right), \qquad k = 1, 2, \ldots [3\gamma/2], \quad k \in \mathcal{N}, \tag{7.1}$$

where the notation $[\mu]$ represents the largest integer less than μ. The masses of these bound states are

$$m_k = 2\,M \sin\left(\frac{\pi k}{3\gamma}\right), \tag{7.2}$$

where M is the soliton mass (2.7). The bootstrap will be used to calculate the breather transmission factors and, for the lowest mass breathers ($k = 1$), it states

$$c_1{}^{\bar{a}a}\, T^{b_1}(\theta)\, \delta_\alpha^\beta = \sum_e T^2{}_{\bar{e}\alpha}^{\bar{a}\gamma}\,(\theta - i(\pi/2 - \pi/3\gamma))\; T^1{}_{e\gamma}^{a\beta}\,(\theta + i(\pi/2 - \pi/3\gamma))\; c_1{}^{\bar{e}e}. \tag{7.3}$$

JHEP07(2007)001

The ratios of the couplings can be calculated using the scattering matrices provided in appendix A. For instance using the matrix S^{12}, the couplings calculated at $\theta_{12} = i\pi(1 - 2/3\gamma)$ satisfy

$$\begin{aligned} &c_{i\bar{\imath}}{}^{1}\, c_1{}^{\bar{\imath}i} = -\rho_1^{12}\, (q - q^{-1}), \qquad && c_{i\bar{m}}{}^{1}\, c_1{}^{\bar{m}i} = 0, \qquad i, m = 1, 2, 3, \quad m \neq i, \\ &c_{j\bar{\jmath}}{}^{1}\, c_1{}^{\bar{l}l} = \rho_1^{12}\, (q - q^{-1})\, e^{i\pi/3}, && c_{j\bar{\jmath}}{}^{1}\, c_1{}^{\bar{k}k} = \rho_1^{12}\, (q - q^{-1})\, e^{-i\pi/3}, \\ &\qquad j = 1, 2, 3 \quad k, l \neq j \qquad (k = j + 1, l = j + 2) \mod (3), \end{aligned}$$

where ρ_1^{12} is the scalar function ρ^{12} calculated when $\theta_{12} = i\pi(1 - 2/3\gamma)$. Consequently, the coupling ratios appearing in (7.3) are

$$\frac{c_1{}^{\bar{3}3}}{c_1{}^{\bar{1}1}} = \frac{c_1{}^{\bar{1}1}}{c_1{}^{\bar{2}2}} = \frac{c_1{}^{\bar{2}2}}{c_1{}^{\bar{3}3}} = -e^{i\pi/3}, \qquad \frac{c_1{}^{3\bar{3}}}{c_1{}^{1\bar{1}}} = \frac{c_1{}^{1\bar{1}}}{c_1{}^{2\bar{2}}} = \frac{c_1{}^{2\bar{2}}}{c_1{}^{3\bar{3}}} = -e^{-i\pi/3}. \tag{7.4}$$

Clearly, identical coupling ratios are obtained using the matrix S^{21} instead of S^{12}. Using the transmission matrices (5.9) and (5.10), with the scaling functions g^1, g^2 given by (6.7), (6.9), the transmission factors for the lightest breathers are

$$T_1^{b_1}(\theta) = e^{-i\pi/3} \frac{\sinh\left(\frac{\theta - \eta}{2} - \frac{i\pi}{6}\right)}{\sinh\left(\frac{\theta - \eta}{2} + \frac{i\pi}{6}\right)}, \qquad T_2^{b_1}(\theta) = e^{i\pi/3} \frac{\sinh\left(\frac{\theta - \eta}{2} + \frac{i2\pi}{3}\right)}{\sinh\left(\frac{\theta - \eta}{2} + \frac{i\pi}{3}\right)}. \tag{7.5}$$

Notice that, as was the case for sine-Gordon [8], the transmission factors for the lightest breathers appear to depend on the coupling constant β only via the parameter η. Nevertheless, one might expect, in the classical limit $\beta \to 0$, that the parameter represented by η appearing in (7.5) would tend to the classical Lagrangian parameter. For this reason, as mentioned previously, the same notation has been used for this parameter regardless of the context.

Consider now the classical problem of finding the transmission coefficients for the linearized version of the jump-defect problem. Following the procedure adopted in [21] for the affine Toda field theory restricted to a half line, the bulk fields ϕ and ψ can be expanded in power series in β as follows,

$$\phi = \sum_{k=-1}^{\infty} \beta^k\, \phi^{(k)}, \qquad \psi = \sum_{k=-1}^{\infty} \beta^k\, \psi^{(k)}.$$

The fields $\phi^{(0)}$ and $\psi^{(0)}$ represent the small coupling limit, namely small perturbations around the background represented by the fields $\phi^{(-1)}$ and $\psi^{(-1)}$. The field $\phi^{(0)}, \psi^{(0)}$ satisfy the linearized version of the equations of motion and the defect conditions. Since the background represents the ground state, it is supposed to have minimal energy and to be time-independent. Any static configuration (2.16), as well as the choice $(\phi^{(-1)}, \psi^{(-1)}) = (0, 0)$, satisfy these requirements. Then, the equations of motion and defect conditions for the fields $\phi^{(0)}, \psi^{(0)}$ become, respectively

$$\begin{aligned} \partial_t^2 \phi^{(0)} - \partial_x^2 \phi^{(0)} &= -m^2 \sum_{i=0}^{r} \alpha_i \alpha_i^T \cdot \phi^{(0)} = -\mathcal{M}^2 \phi^{(0)}, \qquad x < 0, \\ \partial_t^2 \psi^{(0)} - \partial_x^2 \psi^{(0)} &= -m^2 \sum_{i=0}^{r} \alpha_i \alpha_i^T \cdot \psi^{(0)} = -\mathcal{M}^2 \psi^{(0)}, \qquad x > 0, \end{aligned} \tag{7.6}$$

$$\partial_x\phi^{(0)} - E\partial_t\phi^{(0)} - D\partial_t\psi^{(0)} + \frac{m\sigma}{4}\sum_{i=0}^{r}(\alpha_i\cdot D^T)\left[(\alpha_i\cdot D^T)\phi^{(0)} + (\alpha_i\cdot D)\psi^{(0)}\right] \tag{7.7}$$

$$+\frac{m}{4\sigma}\sum_{i=0}^{r}(\alpha_i\cdot D)\left[(\alpha_i\cdot D)\phi^{(0)} - (\alpha_i\cdot D)\psi^{(0)}\right] = 0, \quad x=0,$$

$$\partial_x\psi^{(0)} - D^T\partial_t\phi^{(0)} + E\partial_t\psi^{(0)} - \frac{m\sigma}{4}\sum_{i=0}^{r}(\alpha_i\cdot D)\left[(\alpha_i\cdot D^T)\phi^{(0)} + (\alpha_i\cdot D)\psi^{(0)}\right]$$

$$+\frac{m}{4\sigma}\sum_{i=0}^{r}(\alpha_i\cdot D)\left[(\alpha_i\cdot D)\phi^{(0)} - (\alpha_i\cdot D)\psi^{(0)}\right] = 0, \quad x=0,$$

where $\mathcal{M}$ represents the mass matrix. A solution of the equations of motion (7.6) is

$$\begin{aligned} \phi^{(0)} &= \sum_{k=1}^{r}\rho_k\left(e^{ib_kx} + R_k e^{-ib_kx}\right)e^{-ia_kt}, && x<0, \\ \psi^{(0)} &= \sum_{k=1}^{r}\rho_k T_k e^{i(b_kx-a_kt)}, && x>0. \end{aligned} \tag{7.8}$$

The vector ρ_k, in order to satisfy the equations (7.6), has to be an eigenvector of the matrix $\mathcal{M}$, namely

$$\mathcal{M}^2\rho_k = (a_k^2 - b_k^2)\rho_k = m_k^2\rho_k, \qquad m_k = 2\,m\sin\left(\frac{k\pi}{h}\right), \qquad k=1,\dots,r. \tag{7.9}$$

By contrast with the half-line case discussed in [21], the vectors ρ_k do not diagonalise the defect conditions (7.7), because of the presence of the matrix D. Therefore, explicit expressions for these vectors are required in order to find the coefficients $R_k,\ T_k$ appearing in (7.7). Bearing in mind the expression (2.6) for the soliton solutions, the vectors ρ_k can be written as follows

$$\rho_k = -\sum_{l=0}^{r}\alpha_l\,\omega^{lk}, \qquad \omega = e^{2\pi i/h}, \qquad k=1,\dots,r. \tag{7.10}$$

It can be verified easily that these vectors satisfy (7.9).

Inserting (7.8) into the linearized defect conditions (7.7), two expressions containing the unknown coefficients $R_k,\ T_k$ are obtained. Multiplying them on the left hand side by $\rho_k^\dagger$, and making use of (2.12) and (7.10), leads to

$$\begin{aligned} &R_k[(-ib_k + ia_k + \sigma + 1/\sigma)(2-\omega^{-k}-\omega^k) - 2ia_k(1-\omega^k)] \\ &\quad + T_k[2ia_k(1-\omega^k) - (2-\omega^{-k}-\omega^k)(\sigma\omega^{-k} + 1/\sigma)] + ib_k(2-\omega^{-k}-\omega^k) \\ &\qquad + (\sigma+1/\sigma)(2-\omega^{-k}-\omega^k) + ia_k(\omega^k-\omega^{-k}) = 0, \quad k=1,\dots,r, \\ &R_k[2ia_k(1-\omega^{-k}) + (\sigma\omega^{-k} + 1/\sigma)(2-\omega^k-\omega^{-k})] \\ &\quad + T_k[ib_k(2-\omega^{-k}-\omega^k) + ia_k(\omega^{-k}-\omega^k) - (\sigma+1/\sigma)(2-\omega^k-\omega^{-k})] \\ &\qquad + 2ia_k(1-\omega^{-k}) + (\sigma\omega^k + 1/\sigma)(2-\omega^{-k}-\omega^k) = 0, \quad k=1,\dots,r. \end{aligned}$$

After a little bit of algebra, and setting $a_k = m_k \cosh\theta$, $b_k = m_k \sinh\theta$, the reflection and transmission coefficients are found to be

$$R_k = 0, \qquad T_k = \frac{ie^{-\theta}m_k - \sigma(1-\omega^{-k})}{ie^{\theta}m_k + \sigma(1-\omega^{k})}, \qquad k = 1,\dots,r. \tag{7.11}$$

It can be easily verified for the a_2 affine Toda field theory, setting $\sigma \equiv e^{-\eta}$, that the transmission coefficients (7.11) ($r = 1, 2$) coincide with the expressions (7.5) for the transmission factors for the lightest breathers. Moreover, as expected, the reflection factors turn out to be zero. This result, given that no perturbative calculations are available to suggest otherwise, provides a further justification for the choice made in (6.12) for the constant Δ.

8. On defects and solutions of the triangle equations

The investigation of the triangle equations for the a_2 affine Toda field theory reveals several possible candidates for the transmission matrix describing the interaction between solitons and the purely transmitting defects. In previous sections, one of these solutions has been chosen to be the 'correct' matrix describing the scattering between solitons and the jump-defect discussed, classically, in section 2. Some evidence to support this choice have been provided. However, for reasons of completeness, some words are also due concerning the other solutions listed in appendix B.

As mentioned in section (2), the choice of the 'clockwise' cyclic permutation of the simple roots was arbitrary and made simply in order to give a specific expression for the D matrix. In fact, as was already pointed out, the other possibility, using the 'anticlockwise' permutation, was chosen in [5]. It turns out that if the alternative choice had been adopted from the start, the corresponding transmission matrix for solitons would have been given by (B.35), instead of (B.23), and it would have been the soliton transmission matrix that had the zeros corresponding to classical selection rules. Applying the bootstrap procedure to (B.35) the resulting transmission matrix for the antisolitons would be found to have no zero components. As a consequence, for this alternative choice it is the matrix for the solitons that mirrors the classical selection rules for the delays of solitons and antisolitons passing through the defect. Thus, reversing the sense of the permutation has the effect of maintaining the asymmetry but interchanging the roles of solitons and antisolitons. Similar arguments to those used to constrain the overall factor $g^1(\theta)$ can be applied with the alternative choice of permutation leading to a suitable overall scalar function $\tilde{g}^1$. The transmission factors for the lightest breathers can be calculated and, provided a suitable choice for the single independent parameter appearing in the transmission matrix is made, it can be verified they coincide with (7.5), as was to be expected.

It should be noted that an alternative setting for the jump-defect problem is also possible. Classically, the distinction between the two settings turned out to be important in the process of calculating of conserved charges [5]. In fact, according to which setting is chosen, only the even or odd spin charges are conserved (apart from the spin ±1 charges that correspond to energy and momentum). The defect conditions for the alternative

JHEP07(2007)001

framework are

$$\begin{aligned} \partial_x\phi + E^T\partial_t\phi - D\partial_t\psi + \partial_\phi\mathcal{B} = 0 \qquad x = 0, \\ \partial_x\psi - D^T\partial_t\phi - E^T\partial_t\psi - \partial_\psi\mathcal{B} = 0 \qquad x = 0. \end{aligned} \tag{8.1}$$

with D substituted by $-D$ in the defect potential (2.10), and $D + D^T = -2$, $E = 1 + D^T$. An explicit expression for the matrix D, provided a clockwise permutation is chosen, is given by

$$D = 2\sum_{a=1}^{r} w_a\,(w_{a+1} - w_a)^T\,. \tag{8.2}$$

In the quantum context the transmission matrix describing this jump-defect framework is given by (B.36). Alternatively, choosing the anticlockwise permutation the correct transmission matrix would be given by (B.24). All the computations performed in this article can be repeated for this alternative framework without any problems.

The remaining solutions of the triangle equations do not seem to be relevant for the defect. Some of them fail to be invertible implying they will never satisfy (6.2), and others fail to fit the pattern implied by the functional arguments presented in section (4).

9. Conclusion

The purpose of this article has been to extend previous work devoted to the sine-Gordon theory with a defect. During the analysis several intriguing results have emerged. One of these, and perhaps the most interesting, is the appearance of an unstable soliton-defect bound state within a band of couplings that does not include a neighbourhood of the classical limit. In a way, this is natural for the a_2 model because solitons cannot be absorbed by the defect within the classical field theory; though a logical alternative would have been a complete absence of unstable states. The next step to take will be to examine the a_n models in sufficient detail to be able to determine the pattern of bound states accompanying defect-soliton scattering. One of the first steps will be to analyse a defect interacting with solitons whose topological charges are described by weights in the six-dimensional representation (corresponding to the centre spot in the Dynkin diagram) of a_3. This is the first occasion where a classical model has missing solutions (four of them, corresponding to four particular weights in the **6**), and it will be interesting to see if there is a mechanism to generate states corresponding to them in the quantum theory.

Another intriguing feature is the manner by which the quantum field theory with a defect chooses to implement the classical selection rules governing the transitions between different topological charges that are permitted by the defect. In all cases (whether it be the choice of setting or permutation describing the defect in the Lagrangian), there is an imbalance represented by the curious asymmetry between the behaviour of solitons and the behaviour of antisolitons represented typically by (5.9) and (5.10). Some difference between soliton and antisoliton behaviour was to be expected owing to the explicit breaking by the defect of parity and time-reversal but the way the difference reveals itself is quite peculiar.

JHEP07(2007)001

A further interesting fact concerns the matrix E (or equivalently $(1-D)$). This matrix is an ingredient of the defect part of the classical Lagrangian and determined, in the first place, by insisting upon classical integrability. However, in section (4) it has been shown how it is alternatively specified by examining the bootstrap in the functional integral context.

It was pointed out in [8] that it appears to be perfectly consistent to allow several defects, or indeed to allow defects to move with independent velocities. This part of sine-Gordon story has not been explored yet for the other affine Toda theories and must be deferred for the moment.

One final remark. In most respects, members of the full set of affine Toda field models share similar features, with such differences as there are attributable to their differing root data. In gross terms, a feature of one of them is a feature of the others. However, the classical analysis of integrable defects has only revealed (so far) the possibility of defects within the a_r series of models [5]. On the other hand, all models in the imaginary coupling regime have an S-matrix to describe the scattering of solitons, and one would expect within each of these models a wide variety of infinite-dimensional solutions to the triangle equations. It remains to be seen if any of these solutions can be interpreted as soliton-defect scattering though it would be surprising if such was not the case. Pushing the analysis in this direction may shed some light on the existence (or otherwise) of a wider class of integrable defect.

Acknowledgments

One of us (CZ) is supported by a postdoctoral fellowship SPM 06-13 supplied by the Centre National de la Recherche Scientifique (CNRS), and thanks the Japan Society for the Promotion of Science for a Postdoctoral Fellowship when part of this work was performed; the other (EC) thanks the Yukawa Institute for Theoretical Physics, Kyoto University for hospitality on a number of occasions coinciding with several stages of this work. The work has also been supported in part by EUCLID (a European Commission Research Training Network, contract number HPRN-CT-2002-00325).

A. S-matrices for the a_2 affine Toda field theory

For the a_2 affine Toda field theory, apart from the scattering matrix S^{11} already described in section (3), the matrices S^{12} and S^{21} are also used in the present article. Consider the following triple product

$$\begin{aligned} A^1_l(\theta_1)A^2_{\bar{k}}(\theta_2) \equiv\ & A^1_l(\theta_1)[c_{\bar{k}}{}^{ij}\, A^1_i(\theta_2 - i\pi/3)A^1_j(\theta_2 + i\pi/3) \\ & \qquad + c_{\bar{k}}{}^{ji}\, A^1_j(\theta_2 - i\pi/3)A^1_i(\theta_2 + i\pi/3)], \qquad l^2_{\bar{k}} = l_i + l_j, \end{aligned} \tag{A.1}$$

where $l^2_{\bar{k}} \equiv l_{\bar{k}} = -l_k$ $(k = 1, 2, 3)$ and the value of the couplings is given in (3.10). Then, making use of (3.5), the non-zero components of the matrix S^{12} are given by

$$
\begin{aligned}
A^1_j(\theta_1)A^2_{\bar{j}}(\theta_2) &= \sum_{k=1}^{3} S^{12}{}^{\bar{k}\,k}_{j\,\bar{j}}(\theta_{12})A^2_{\bar{k}}(\theta_2)A^1_k(\theta_1), \\
A^1_j(\theta_1)A^2_{\bar{k}}(\theta_2) &= S^{12}{}^{\bar{k}\,j}_{j\,\bar{k}}(\theta_{12})A^2_{\bar{k}}(\theta_2)A^1_j(\theta_1), \quad j \neq k,
\end{aligned} \tag{A.2}
$$

with

$$
\begin{aligned}
S^{12}{}^{\bar{j}\,j}_{j\,\bar{j}}(\theta_{12}) &= \quad \rho^{12}(\theta_{12}) \left(x_{12}(-q)^{1/2} - x_{12}^{-1}(-q)^{-1/2} \right), \\
S^{12}{}^{\bar{k}\,k}_{j\,\bar{j}}(\theta_{12}) &= \quad \rho^{12}(\theta_{12}) \left(q - q^{-1} \right) \begin{cases} x_{12}^{1/3}(-q)^{\;1/2}, & k = j-1 \mod (3) \\ x_{12}^{-1/3}(-q)^{-1/2}, & k = j+1 \mod (3) \end{cases} \\
S^{12}{}^{\bar{k}\,j}_{j\,\bar{k}}(\theta_{12}) &= -\rho^{12}(\theta_{12}) \left(x_{12}(-q)^{3/2} - x_{12}^{-1}(-q)^{-3/2} \right), \quad j \neq k,
\end{aligned} \tag{A.3}
$$

where the scalar mulitplier ρ^{12} is

$$
\rho^{12}(\theta_{12}) = \left(x_{12}(-q)^{-1/2} - x_{12}^{-1}(-q)^{1/2} \right) \rho^{11}(\theta_{12} + \pi i/3)\, \rho^{11}(\theta_{12} - \pi i/3).
$$

The non-zero elements of the matrix S^{21} are equal to the elements described in (A.3) for the matrix S^{12} with $\rho^{12} = \rho^{21}$, except for the $S^{21}{}^{k\,\bar{k}}_{\bar{j}\,j}$ elements, for which the index $k = j-1$ has to be replaced by $k = j+1$, and vice versa. Such a small difference turns out to be relevant in the calculation of the transmission factors for the lightest breathers performed in section (7). A detailed investigation of the a_2 affine Toda field theory, including bound states and scattering processes, can be found in [20].

B. Solutions of the triangle equations for the a_2 model

A classification of the possible solutions of the Yang-Baxter equation for purely transmitting defects (5.1) will be provided in this appendix. As already explained in section (5), because of the topological charge conservation, the ansatz for the elements of the transmission matrix is supplied by (5.2). For the analysis performed in this appendix, it is useful to assign a different letter to each entries of the transmission matrix to avoid the use of many indexes. The notation chosen is the following

$$
T^{j\beta}_{i\alpha}(\theta) \equiv \begin{pmatrix} A^\beta_\alpha(\theta) & K^\beta_\alpha(\theta) & V^\beta_\alpha(\theta) \\ J^\beta_\alpha(\theta) & B^\beta_\alpha(\theta) & I^\beta_\alpha(\theta) \\ W^\beta_\alpha(\theta) & L^\beta_\alpha(\theta) & C^\beta_\alpha(\theta) \end{pmatrix} = \begin{pmatrix} a_\alpha(\theta)\,\delta^\beta_\alpha & k_\alpha(\theta)\,\delta^{\beta-\alpha}_\alpha & v_\alpha(\theta)\,\delta^{\beta+\alpha_0}_\alpha \\ j_\alpha(\theta)\,\delta^{\beta+\alpha_1}_\alpha & b_\alpha(\theta)\,\delta^\beta_\alpha & i_\alpha(\theta)\,\delta^{\beta-\alpha_2}_\alpha \\ w_\alpha(\theta)\,\delta^{\beta-\alpha_0}_\alpha & l_\alpha(\theta)\,\delta^{\beta+\alpha_2}_\alpha & c_\alpha(\theta)\,\delta^\beta_\alpha \end{pmatrix}.
$$

In addition, the following short notation will be adopted

$$
(q\,x_{12} - q^{-1}\,x_{12}^{-1}) \equiv a, \qquad (x_{12} - x_{12}^{-1}) \equiv b, \qquad (q - q^{-1}) \equiv c, \tag{B.1}
$$

with

$$
x_{12} = \frac{x_1}{x_2}, \qquad x_j = e^{3\gamma\theta/2}, \qquad q = -e^{-i\pi\gamma}.
$$

As a starting assumption, all entries of the transmission matrix are supposed to be different from zero. Expression (5.1) provides several relations involving the elements of the T matrix, which can be gathered into five groups. For each group of relations, examples will be provided. The notation A_1, A_2 etc. will be used to indicate the entries $A(\theta_1)$, $A(\theta_2)$, respectively.

- Group 1

$$A_1 A_2 = A_2 A_1, \tag{B.2}$$

and eight more equations, one for each entry of the T matrix. For the diagonal entries, this kind of relation is automatically satisfied, while for the other entries they state that the ratios

$$\frac{k_{\alpha+\alpha_1}}{k_\alpha}, \qquad \frac{j_{\alpha+\alpha_1}}{j_\alpha}, \qquad \frac{v_{\alpha+\alpha_0}}{v_\alpha}, \qquad \frac{w_{\alpha+\alpha_0}}{w_\alpha}, \qquad \frac{i_{\alpha+\alpha_2}}{i_\alpha}, \qquad \frac{l_{\alpha+\alpha_2}}{l_\alpha}, \tag{B.3}$$

are independent of rapidity.

- Group 2

$$\begin{aligned} b(A_1B_2 - B_2A_1) &= c(J_2K_1x_{12}^{-1/3} - J_1K_2x_{12}^{1/3}), \\ b(B_1A_2 - A_2B_1) &= c(K_2J_1x_{12}^{1/3} - K_1J_2x_{12}^{-1/3}), \\ c\,x_{12}^{1/3}(B_1A_2 - B_2A_1) &= b(J_2K_1 - K_1J_2), \\ c\,x_{12}^{-1/3}(A_1B_2 - A_2B_1) &= b(K_2J_1 - J_1K_2), \end{aligned} \tag{B.4}$$

and another two similar series of four relations involving the elements A, C, W, V and B, C, I, L, respectively. The first two expressions in (B.4) force the ratio $(x^{-2/3}\ k_\alpha)/j_\alpha$ to be independent of rapidity. The remaining two expressions are not independent since their sum turns out to be zero. Therefore, only one expression in (B.4) has still to be analyzed. Similar conclusions can be drawn from the other two series of relations. In the end the constraints state that

$$\frac{k_\alpha}{j_\alpha}x^{-2/3}, \qquad \frac{w_\alpha}{v_\alpha}x^{-2/3}, \qquad \frac{i_\alpha}{l_\alpha}x^{-2/3}, \tag{B.5}$$

are independent of rapidity and three expressions remain to be analyzed. The latter will be kept on one side to be discussed at the end of this section.

- Group 3

$$\begin{aligned} aA_1K_2 - bK_2A_1 &= c\,x_{12}^{-1/3}A_2K_1, & aK_1A_2 - bA_2K_1 &= c\,x_{12}^{1/3}K_2A_1, \\ aA_2J_1 - bJ_1A_2 &= c\,x_{12}^{-1/3}A_1J_2, & aJ_2A_1 - bA_1J_2 &= c\,x_{12}^{1/3}J_1A_2, \end{aligned} \tag{B.6}$$

$$\begin{aligned} aA_1V_2 - bV_2A_1 &= c\,x_{12}^{1/3}A_2V_1, & aV_1A_2 - bA_2V_1 &= c\,x_{12}^{-1/3}V_2A_1, \\ aA_2W_1 - bW_1A_2 &= c\,x_{12}^{1/3}A_1W_2, & aW_2A_1 - bA_1W_2 &= c\,x_{12}^{-1/3}W_1A_2. \end{aligned} \tag{B.7}$$

The relations (B.6) are satisfied if $a_{\alpha+\alpha_1}/a_\alpha = q$ and the two ratios $a_\alpha\, x^{-2/3}/k_\alpha$ and $a_\alpha\, x^{-4/3}/j_\alpha$ are independent of rapidity; or, if $a_{\alpha+\alpha_1}/a_\alpha = 1/q$ and the ratios $a_\alpha\, x^{4/3}/k_\alpha$, $a_\alpha\, x^{2/3}/j_\alpha$ are independent of rapidity. In short, these solutions are summarized as follows

$$\frac{a_{\alpha+\alpha_1}}{a_\alpha} = q, \quad \frac{a_\alpha}{k_\alpha}x^{-2/3}, \quad \frac{a_\alpha}{j_\alpha}x^{-4/3} \quad \text{or} \quad \frac{a_{\alpha+\alpha_1}}{a_\alpha} = \frac{1}{q}, \quad \frac{a_\alpha}{k_\alpha}x^{4/3}, \quad \frac{a_\alpha}{j_\alpha}x^{2/3}. \tag{B.8}$$

Similarly, for the expressions (B.7) the solutions are

$$\frac{a_{\alpha+\alpha_0}}{a_\alpha} = \frac{1}{q}, \quad \frac{a_\alpha}{v_\alpha}x^{-4/3}, \quad \frac{a_\alpha}{w_\alpha}x^{-2/3} \quad \text{or} \quad \frac{a_{\alpha+\alpha_0}}{a_\alpha} = q, \quad \frac{a_\alpha}{v_\alpha}x^{2/3}, \quad \frac{a_\alpha}{w_\alpha}x^{4/3}. \tag{B.9}$$

In group 3 there are further relations, among which there are eight involving the elements B and K, J, L, I and another eight involving C and V, W, L, I. Using the same notation as (B.8) and (B.9), the constraints expressed by the other relations of this group, which are not listed here, are

$$\frac{b_{\alpha+\alpha_1}}{b_\alpha} = q, \quad \frac{b_\alpha}{k_\alpha}x^{4/3}, \quad \frac{b_\alpha}{j_\alpha}x^{2/3} \quad \text{or} \quad \frac{b_{\alpha+\alpha_1}}{b_\alpha} = \frac{1}{q}, \quad \frac{b_\alpha}{k_\alpha}x^{-2/3}, \quad \frac{b_\alpha}{j_\alpha}x^{-4/3} \tag{B.10}$$

$$\frac{b_{\alpha+\alpha_2}}{b_\alpha} = q, \quad \frac{b_\alpha}{i_\alpha}x^{-2/3}, \quad \frac{b_\alpha}{l_\alpha}x^{-4/3} \quad \text{or} \quad \frac{b_{\alpha+\alpha_2}}{b_\alpha} = \frac{1}{q}, \quad \frac{b_\alpha}{i_\alpha}x^{4/3}, \quad \frac{b_\alpha}{l_\alpha}x^{2/3} \tag{B.11}$$

and

$$\frac{c_{\alpha+\alpha_0}}{c_\alpha} = q, \quad \frac{c_\alpha}{v_\alpha}x^{-4/3}, \quad \frac{c_\alpha}{w_\alpha}x^{-2/3} \quad \text{or} \quad \frac{c_{\alpha+\alpha_0}}{c_\alpha} = \frac{1}{q}, \quad \frac{c_\alpha}{v_\alpha}x^{2/3}, \quad \frac{c_\alpha}{w_\alpha}x^{4/3} \tag{B.12}$$

$$\frac{c_{\alpha+\alpha_2}}{c_\alpha} = q, \quad \frac{c_\alpha}{i_\alpha}x^{4/3}, \quad \frac{c_\alpha}{l_\alpha}x^{2/3} \quad \text{or} \quad \frac{c_{\alpha+\alpha_2}}{c_\alpha} = \frac{1}{q}, \quad \frac{c_\alpha}{i_\alpha}x^{-2/3}, \quad \frac{c_\alpha}{l_\alpha}x^{-4/3}. \tag{B.13}$$

Clearly, the results of the group 3 can be gathered in turn into three subgroups, which will be called $3A$, $3B$ and $3C$ because of the fact that their relations incorporate the diagonal entries A, B or C of the T matrix. Note that each of these subgroups provides four possible different solutions, according to the combination chosen. For instance, for the subgroup $3A$ it is possible to choose the first expression in (B.8) together with the first expression in (B.9), or the second one. These are already two different combinations. Similarly, starting with the second expression in (B.8). Note also that each subgroup provides a complete understanding of the dependence of the diagonal elements of the T matrix with respect to the simple roots, and therefore with respect to a general vector $\alpha = m\,\alpha_1 + n\,\alpha_2$. In fact, it is possible to conclude that the ratios $a_{\alpha+\alpha_2}/a_\alpha$, $b_{\alpha+\alpha_0}/b_\alpha$ and $c_{\alpha+\alpha_1}/c_\alpha$ can only be equal to 1 or $1/q^2$ or q^2. This piece of information will be relevant for the analysis of the next group of equations.

- Group 4

$$\begin{aligned}
b(A_1I_2 - I_2A_1) &= c\,x_{12}^{1/3}(J_2V_1 - J_1V_2); & b(I_1A_2 - A_2I_1) &= c\,x_{12}^{-1/3}(V_2J_1 - V_1J_2),\\
b(J_1V_2 - V_2J_1) &= (x_{12}^{1/3}A_2I_1 - x_{12}^{-1/3}A_1I_2); & b(V_1J_2 - J_2V_1) &= c(x_{12}^{-1/3}I_2A_1 - x_{12}^{1/3}I_1A_2),\\
b(L_1A_2 - A_2L_1) &= c\,x_{12}^{1/3}(K_2W_1 - K_1W_2); & b(A_1L_2 - L_2A_1) &= c\,x_{12}^{-1/3}(W_2K_1 - W_1K_2),\\
b(K_1W_2 - W_2K_1) &= c(x_{12}^{1/3}L_2A_1 - x_{12}^{-1/3}L_1A_2); & b(W_1K_2 - K_2W_1) &= c(x_{12}^{-1/3}A_2L_1 - x_{12}^{1/3}A_1L_2),
\end{aligned} \tag{B.14}$$

with sixteen other similar relations, eight involving the element B together with all the off diagonal elements of the T matrix, and eight involving the C element together with all the off diagonal entries. The first constraint provided by all these equations is the following

$$\frac{a_{\alpha+\alpha_2}}{a_\alpha} = \frac{b_{\alpha+\alpha_0}}{b_\alpha} = \frac{c_{\alpha+\alpha_1}}{c_\alpha} = 1. \tag{B.15}$$

In other words, the other possibilities mentioned previously are not permitted, since they contradict (B.5). This observation allows a reduction in the number of possible combinations of solutions in each group $3A$, $3B$ and $3C$ from four to two. At this stage, for pursuing the analysis of the triangular relations, it is useful to adopt the following notation for the ratios independent of rapidity appearing in (B.8)-(B.13):

$$\frac{a_\alpha}{p_\alpha} x^{\pm\epsilon_p/3} = \frac{1}{h_{ap}(\alpha)} \frac{t_a}{t_p}, \quad \frac{b_\alpha}{p_\alpha} x^{\pm\epsilon_p/3} = \frac{1}{h_{bp}(\alpha)} \frac{t_b}{t_p}, \quad \frac{c_\alpha}{p_\alpha} x^{\pm\epsilon_p/3} = \frac{1}{h_{cp}(\alpha)} \frac{t_c}{t_p}, \quad \epsilon_p = 2, 4, \tag{B.16}$$

where h_{kj} are exponential functions, t_k are constants and p_α stands for one of the off-diagonal entries of the T matrix appearing in expressions (B.8)-(B.13). With this notation, the constraints provided by the relations in group 4 can be summarized as follows.

Consider the relations in the subgroup $3A$, namely (B.8) and (B.9). If the combination

$$\frac{a_{\alpha+\alpha_1}}{a_\alpha} = q, \qquad \frac{a_{\alpha+\alpha_0}}{a_\alpha} = \frac{1}{q}, \tag{B.17}$$

holds, then

$$\frac{h_{aj}(\alpha_0)}{h_{av}(\alpha_1)} = q^2, \qquad \frac{h_{ak}(\alpha_0)}{h_{aw}(\alpha_1)} = 1, \qquad h_{ak}(\alpha) h_{aw}(\alpha+\alpha_1) t_k t_w = h_{al}(\alpha) t_l t_a.$$

On the other hand, if

$$\frac{a_{\alpha+\alpha_1}}{a_\alpha} = \frac{1}{q}, \qquad \frac{a_{\alpha+\alpha_0}}{a_\alpha} = q, \tag{B.18}$$

then

$$\frac{h_{aj}(\alpha_0)}{h_{av}(\alpha_1)} = 1, \qquad \frac{h_{ak}(\alpha_0)}{h_{aw}(\alpha_1)} = \frac{1}{q^2}, \qquad h_{aj}(\alpha) h_{av}(\alpha-\alpha_1) t_j t_v = h_{ai}(\alpha) t_i t_a.$$

Similarly, consider the relations in the subgroup $3B$. If

$$\frac{b_{\alpha+\alpha_1}}{b_\alpha} = \frac{1}{q}, \qquad \frac{b_{\alpha+\alpha_2}}{b_\alpha} = q, \tag{B.19}$$

then

$$\frac{h_{bl}(\alpha_1)}{h_{bj}(\alpha_2)} = q^2, \qquad \frac{h_{bi}(\alpha_1)}{h_{bk}(\alpha_2)} = 1, \qquad h_{bi}(\alpha) h_{bk}(\alpha+\alpha_2) t_i t_k = h_{bv}(\alpha) t_v t_b.$$

On the other hand, if

$$\frac{b_{\alpha+\alpha_1}}{b_\alpha} = q, \qquad \frac{b_{\alpha+\alpha_2}}{b_\alpha} = \frac{1}{q}, \tag{B.20}$$

then

$$\frac{h_{bl}(\alpha_1)}{h_{bj}(\alpha_2)} = 1, \qquad \frac{h_{bi}(\alpha_1)}{h_{bk}(\alpha_2)} = \frac{1}{q^2}, \qquad h_{bl}(\alpha) h_{bj}(\alpha-\alpha_2) t_l t_j = h_{bw}(\alpha) t_w t_b.$$

JHEP07(2007)001

Finally, looking at the relations in the subgroup $3C$. If

$$\frac{c_{\alpha+\alpha_0}}{c_\alpha} = q, \qquad \frac{c_{\alpha+\alpha_2}}{c_\alpha} = \frac{1}{q}, \tag{B.21}$$

then

$$\frac{h_{cv}(\alpha_2)}{h_{cl}(\alpha_0)} = q^2, \qquad \frac{h_{cw}(\alpha_2)}{h_{ci}(\alpha_0)} = 1, \qquad h_{ci}(\alpha)h_{cw}(\alpha+\alpha_2)t_i t_w = h_{cj}(\alpha)t_j t_c.$$

Instead, if

$$\frac{c_{\alpha+\alpha_0}}{c_\alpha} = \frac{1}{q}, \qquad \frac{c_{\alpha+\alpha_2}}{c_\alpha} = q, \tag{B.22}$$

then

$$\frac{h_{cv}(\alpha_2)}{h_{cl}(\alpha_0)} = 1, \qquad \frac{h_{cw}(\alpha_2)}{h_{ci}(\alpha_0)} = \frac{1}{q^2}, \qquad h_{cl}(\alpha)h_{cv}(\alpha-\alpha_2)t_l t_v = h_{ck}(\alpha)t_k t_c.$$

Finally, the last group of equations is

- Group 5

$$\begin{aligned} aV_1K_2 - bK_2V_1 &= c\,x_{12}^{1/3}V_2K_1, \quad aK_1V_2 - bV_2K_1 = c\,x_{12}^{-1/3}K_2V_1,\\ aW_2J_1 - bJ_1W_2 &= c\,x_{12}^{1/3}W_1J_2, \quad aJ_2W_1 - bW_1J_2 = c\,x_{12}^{-1/3}J_1W_2, \end{aligned}$$

with eight further relations. None of these involve the diagonal terms of the transmission matrix. The constraints provided by them can be summarized as follows

$$\frac{h_{ak}(-\alpha_0)}{h_{av}(\alpha_1)} = \frac{h_{aj}(\alpha_0)}{h_{aw}(-\alpha_1)} = q, \quad \frac{h_{bl}(\alpha_1)}{h_{bk}(-\alpha_2)} = \frac{h_{bi}(-\alpha_1)}{h_{bj}(\alpha_2)} = q, \quad \frac{h_{cw}(-\alpha_2)}{h_{cl}(\alpha_0)} = \frac{h_{cv}(\alpha_2)}{h_{ci}(-\alpha_0)} = q.$$

All these constraints taken together allow eight families of possible solutions that can be written down explicitly. Firstly, note that the functions of the type $h_{ap}(\alpha)$ can be split into the ratios $h_p(\alpha)/h_a(\alpha)$ (see (B.16)), where each

$$h_p(\alpha) = q^{\alpha(m_p\,\alpha_1 + n_p\,\alpha_2)}$$

with m_p and n_p constants. Bearing this in mind, and taking into account that the values of the constants m_p and n_p for the diagonal entries are already known, it is possible to simplify the relations in groups 4 and 5 involving the functions $h_{ap}(\alpha)$, and determine the constants m_p and n_p for the other entries. In order to be as clear as possible, and to avoid the use of a heavy notation, it is sufficient to write down explicitly as an example only one solution for each family. The choice of the explicit solutions, which may be called 'minimal', is motivated by the fact that the solutions relevant for the defect problem lie within this group. Despite that, an example of a complete family of solutions will be provided later, and the use of the term 'minimal' should become clearer.

Rewriting the constants t_p as t_{ij}, indicating their positions in the transmission matrix (for instance, $t_k = t_{12}$), the eight 'minimal' solutions found - one for each family - are ($q^{-1} \equiv Q$):

- $t_{21}\, t_{13} = t_{23}, \qquad \frac{t_{32}\, t_{21}}{t_{22}} = t_{31}, \qquad \frac{t_{32}\, t_{13}}{t_{33}} = t_{12},$

$$\begin{pmatrix} Q^{\alpha\cdot l_1}\, \delta_\alpha^\beta & t_{12}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_1} & t_{13}\, x^{2/3}\, Q^{-\alpha\cdot l_2}\, \delta_\alpha^{\beta+\alpha_0} \\ t_{21}\, x^{2/3}\, Q^{-\alpha\cdot l_3}\, \delta_\alpha^{\beta+\alpha_1} & t_{22}\, Q^{\alpha\cdot l_2}\, \delta_\alpha^\beta & t_{23}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_2} \\ t_{31}\, x^{4/3}\, \delta_\alpha^{\beta-\alpha_0} & t_{32}\, x^{2/3}\, Q^{-\alpha\cdot l_1}\, \delta_\alpha^{\beta+\alpha_2} & t_{33}\, Q^{\alpha\cdot l_3}\, \delta_\alpha^\beta \end{pmatrix} \tag{B.23}$$

- $t_{12}\, t_{31} = t_{32}, \qquad \frac{t_{23}\, t_{21}}{t_{22}} = t_{13}, \qquad \frac{t_{23}\, t_{31}}{t_{33}} = t_{21},$

$$\begin{pmatrix} Q^{-\alpha\cdot l_1}\, \delta_\alpha^\beta & t_{12}\, x^{-2/3}\, Q^{\alpha\cdot l_3}\, \delta_\alpha^{\beta-\alpha_1} & t_{13}\, x^{-4/3}\, \delta_\alpha^{\beta+\alpha_0} \\ t_{21}\, x^{-4/3}\, \delta_\alpha^{\beta+\alpha_1} & t_{22}\, Q^{-\alpha\cdot l_2}\, \delta_\alpha^\beta & t_{23}\, x^{-2/3}\, Q^{\alpha\cdot l_1}\, \delta_\alpha^{\beta-\alpha_2} \\ t_{31}\, x^{-2/3}\, Q^{\alpha\cdot l_2}\, \delta_\alpha^{\beta-\alpha_0} & t_{32}\, x^{-4/3}\, \delta_\alpha^{\beta+\alpha_2} & t_{33}\, Q^{-\alpha\cdot l_3}\, \delta_\alpha^\beta \end{pmatrix} \tag{B.24}$$

These two solutions are the solutions used earlier as the soliton transmission matrices without zero components. The other 'minimal' solutions without zeros follow the same pattern but they are not relevant to the discussion of transmission matrices because they are not invertible. The notation used for them is abbreviated and omits the Kronecker deltas.

- $t_{21}\, t_{13} = t_{23}, \quad \frac{t_{32}\, t_{21}}{t_{22}} = t_{31}, \quad \frac{t_{23}\, t_{31}}{t_{33}} = t_{21},$

$$\begin{pmatrix} Q^{\alpha\cdot l_1} & t_{12}\, x^{4/3} & t_{13}\, x^{2/3}\, Q^{\alpha\cdot l_1} \\ t_{21}\, x^{2/3}\, Q^{-\alpha\cdot l_3} & t_{22}\, Q^{\alpha\cdot l_2} & t_{23}\, x^{4/3}\, Q^{-\alpha\cdot l_3} \\ t_{31}\, x^{4/3}\, Q^{-\alpha\cdot l_3} & t_{32}\, x^{2/3}\, Q^{\alpha\cdot l_2} & t_{33}\, x^2\, Q^{-\alpha\cdot l_3} \end{pmatrix} \tag{B.25}$$

- $t_{12}\, t_{31} = t_{32}, \; \frac{t_{23}\, t_{12}}{t_{22}} = t_{13}, \; \frac{t_{32}\, t_{13}}{t_{33}} = t_{12},$

$$\begin{pmatrix} Q^{-\alpha\cdot l_1} & t_{12}\, x^{-2/3}\, Q^{\alpha\cdot l_3} & t_{13}\, x^{-4/3}\, Q^{\alpha\cdot l_3} \\ t_{21}\, x^{-4/3} & t_{22}\, Q^{-\alpha\cdot l_2} & t_{23}\, x^{-2/3}\, Q^{-\alpha\cdot l_2} \\ t_{31}\, x^{-2/3}\, Q^{-\alpha\cdot l_1} & t_{32}\, x^{-4/3}\, Q^{\alpha\cdot l_3} & t_{33}\, x^{-2}\, Q^{\alpha\cdot l_3} \end{pmatrix} \tag{B.26}$$

- $t_{21}\, t_{13} = t_{23}, \; \frac{t_{23}\, t_{12}}{t_{22}} = t_{13}, \; \frac{t_{32}\, t_{13}}{t_{33}} = t_{12},$

$$\begin{pmatrix} Q^{\alpha\cdot l_1} & t_{12}\, x^{4/3}\, Q^{-\alpha\cdot l_2} & t_{13}\, x^{2/3}\, Q^{-\alpha\cdot l_2} \\ t_{21}\, x^{2/3}\, Q^{\alpha\cdot l_1} & t_{22}\, x^2\, Q^{-\alpha\cdot l_2} & t_{23}\, x^{4/3}\, Q^{-\alpha\cdot l_2} \\ t_{31}\, x^{4/3} & t_{32}\, x^{2/3}\, Q^{\alpha\cdot l_3} & t_{33}\, Q^{\alpha\cdot l_3} \end{pmatrix} \tag{B.27}$$

- $t_{12}\, t_{31} = t_{32}, \; \frac{t_{32}\, t_{21}}{t_{22}} = t_{31}, \; \frac{t_{23}\, t_{31}}{t_{33}} = t_{21},$

$$\begin{pmatrix} Q^{-\alpha\cdot l_1} & t_{12}\, x^{-2/3}\, Q^{-\alpha\cdot l_1} & t_{13}\, x^{-4/3} \\ t_{21}\, x^{-4/3}\, Q^{\alpha\cdot l_2} & t_{22}\, x^{-2}\, Q^{\alpha\cdot l_2} & t_{23}\, x^{-2/3}\, Q^{-\alpha\cdot l_3} \\ t_{31}\, x^{-2/3}\, Q^{\alpha\cdot l_2} & t_{32}\, x^{-4/3}\, Q^{\alpha\cdot l_2} & t_{33}\, Q^{-\alpha\cdot l_3} \end{pmatrix} \tag{B.28}$$

- $t_{21}\,t_{13} = t_{23},\ \frac{t_{23}\,t_{12}}{t_{22}} = t_{13},\ \frac{t_{23}\,t_{31}}{t_{33}} = t_{21},$

$$\begin{pmatrix} Q^{\alpha\cdot l_1} & t_{12}\,x^{4/3}\,Q^{-\alpha\cdot l_2} & t_{13}\,x^{2/3}\,Q^{\alpha\cdot l_1} \\ t_{21}\,x^{2/3}\,Q^{\alpha\cdot l_1} & t_{22}\,x^{2}\,Q^{-\alpha\cdot l_2} & t_{23}\,x^{4/3}\,Q^{\alpha\cdot l_1} \\ t_{31}\,x^{4/3}\,Q^{-\alpha\cdot l_3} & t_{32}\,x^{2/3} & t_{33}\,x^{2}\,Q^{-\alpha\cdot l_3} \end{pmatrix} \tag{B.29}$$

- $t_{12}\,t_{31} = t_{32},\ \frac{t_{32}\,t_{21}}{t_{22}} = t_{31},\ \frac{t_{32}\,t_{13}}{t_{33}} = t_{12},$

$$\begin{pmatrix} Q^{-\alpha\cdot l_1} & t_{12}\,x^{-2/3}\,Q^{-\alpha\cdot l_1} & t_{13}\,x^{-4/3}\,Q^{\alpha\cdot l_3} \\ t_{21}\,x^{-4/3}\,Q^{\alpha\cdot l_2} & t_{22}\,x^{-2}\,Q^{\alpha\cdot l_2} & t_{23}\,x^{-2/3} \\ t_{31}\,x^{-2/3}\,Q^{-\alpha\cdot l_1} & t_{32}\,x^{-4/3}\,Q^{-\alpha\cdot l_1} & t_{33}\,x^{-2}\,Q^{\alpha\cdot l_3} \end{pmatrix}. \tag{B.30}$$

It is not difficult to check the invertibility for these matrices. Consider an infinite-dimensional matrix $\mathcal{A}$ of the general type under consideration:

$$\mathcal{A} = \begin{pmatrix} a_{11}(\alpha)\delta_\alpha^\beta & a_{12}(\alpha)\delta_\alpha^{\beta-\alpha_1} & a_{13}(\alpha)\delta_\alpha^{\beta+\alpha_0} \\ a_{21}(\alpha)\delta_\alpha^{\beta+\alpha_1} & a_{22}(\alpha)\delta_\alpha^\beta & a_{23}(\alpha)\delta_\alpha^{\beta-\alpha_2} \\ a_{31}(\alpha)\delta_\alpha^{\beta-\alpha_0} & a_{32}(\alpha)\delta_\alpha^{\beta+\alpha_2} & a_{33}(\alpha)\delta_\alpha^\beta \end{pmatrix}$$

this is invertible if and only if, for every α,

$$\begin{aligned} &a_{11}(\alpha)\,[a_{22}(\alpha+\alpha_1)a_{33}(\alpha+\alpha_1+\alpha_2) - a_{23}(\alpha+\alpha_1)a_{32}(\alpha+\alpha_1+\alpha_2)] \\ &\quad -a_{12}(\alpha)\,[a_{21}(\alpha+\alpha_1)a_{33}(\alpha+\alpha_1+\alpha_2) - a_{23}(\alpha+\alpha_1)a_{31}(\alpha+\alpha_1+\alpha_2)] \\ &\qquad +a_{31}(\alpha)\,[a_{21}(\alpha+\alpha_1)a_{32}(\alpha+\alpha_1+\alpha_2) - a_{22}(\alpha+\alpha_1)a_{31}(\alpha+\alpha_1+\alpha_2)] \neq 0. \end{aligned}$$

This is similar to the determinant condition for a finite dimensional matrix, but note the shifts in the arguments of the elements. Using this condition, it is easy to demonstrate that only the first two solutions listed above are invertible.

The full family of solutions of which (B.23) is the 'minimal' example is given by

- $t_{21}\,t_{13}\,Q^{\alpha_1\cdot\hat{l}_2} = t_{23}, \qquad \frac{t_{32}\,t_{21}}{t_{22}}\,Q^{\alpha_2\cdot\hat{l}_3} = t_{31}, \qquad \frac{t_{32}\,t_{13}}{t_{33}}\,Q^{\alpha_2\cdot\hat{l}_2} = t_{12},$

$$\begin{pmatrix} Q^{\alpha\cdot l_1}\,\delta_\alpha^\beta & t_{12}\,Q^{\alpha\cdot\hat{l}_3}\,x^{4/3}\,\delta_\alpha^{\beta-\alpha_1} & t_{13}\,x^{2/3}\,Q^{-\alpha\cdot(l_2+\hat{l}_2)}\,\delta_\alpha^{\beta+\alpha_0} \\ t_{21}\,x^{2/3}\,Q^{-\alpha\cdot(l_3+\hat{l}_3)}\,\delta_\alpha^{\beta+\alpha_1} & t_{22}\,Q^{\alpha\cdot l_2}\,\delta_\alpha^\beta & t_{23}\,Q^{\alpha\cdot\hat{l}_1}\,x^{4/3}\,\delta_\alpha^{\beta-\alpha_2} \\ t_{31}\,Q^{\alpha\cdot\hat{l}_2}\,x^{4/3}\,\delta_\alpha^{\beta-\alpha_0} & t_{32}\,x^{2/3}\,Q^{-\alpha\cdot(l_1+\hat{l}_1)}\,\delta_\alpha^{\beta+\alpha_2} & t_{33}\,Q^{\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix} \tag{B.31}$$

where $\hat{l}_p = (\hat{m}_p\,\alpha_1 + \hat{n}_p\,\alpha_2)$ are vectors lying in the weight lattice, such that

$$\alpha_1\cdot\hat{l}_1 = \alpha_2\cdot\hat{l}_3, \qquad \alpha_2\cdot\hat{l}_3 = \alpha_0\cdot\hat{l}_1, \qquad \alpha_0\cdot\hat{l}_3 = \alpha_1\cdot\hat{l}_2. \tag{B.32}$$

It should be noticed that the extra dependence of α in the off-diagonal entries of the matrix (B.31) as compared with corresponding elements in (B.23) does not affect the constraints coming from groups 4 and 5 (dealing with the α dependence), due to (B.32).

Setting $\hat{l}_1 = \hat{l}_2 = \hat{l}_3 = 0$, the solution (B.23) is recovered. It is in this sense that (B.23) is considered a 'minimal' solution.

Finally, it should be pointed out that each solution found can be multiplied by an overall function of θ that cannot be determined by the triangle equations alone.

In addition to all the above solutions, there are others that allow some entries of the T matrix to be set equal to zero. Suppose $K = I = W = 0$, or $J = L = V = 0$, then the previous analysis has to be modified. In this situation, the relations in group 2 state that the following ratios

$$\frac{a_\alpha}{b_\alpha}, \qquad \frac{a_\alpha}{c_\alpha}, \qquad \frac{b_\alpha}{c_\alpha},$$

are independent of rapidity. Concerning group 3, because of the presence of T matrix elements equal to zero, not all constraints listed in (B.8)-(B.13) survive. However, the ones which do are unmodified. Equations in group 4 force relations (B.15) to hold, as before. In addition, if $K = I = W = 0$, they imply the ratios

$$\frac{a_\alpha}{l_\alpha} x^{2/3}, \qquad \frac{b_\alpha}{v_\alpha} x^{2/3}, \qquad \frac{c_\alpha}{j_\alpha} x^{2/3}, \tag{B.33}$$

are independent of rapidity, and only one combination of relations in the subgroups $3A, 3B, 3C$ is allowed, namely

$$\frac{a_{\alpha+\alpha_1}}{a_\alpha} = \frac{1}{q}, \quad \frac{a_{\alpha+\alpha_0}}{a_\alpha} = q, \quad \frac{b_{\alpha+\alpha_1}}{b_\alpha} = q, \quad \frac{b_{\alpha+\alpha_0}}{b_\alpha} = \frac{1}{q}, \quad \frac{c_{\alpha+\alpha_0}}{c_\alpha} = \frac{1}{q}, \quad \frac{c_{\alpha+\alpha_2}}{c_\alpha} = q,$$

with

$$\frac{h_v(\alpha_1)}{h_j(\alpha_0)} = \frac{h_l(\alpha_1)}{h_j(\alpha_2)} = \frac{h_v(\alpha_2)}{h_l(\alpha_0)} = 1.$$

On the other hand, if $J = L = V = 0$, the ratios

$$\frac{a_\alpha}{i_\alpha} x^{-2/3}, \qquad \frac{b_\alpha}{w_\alpha} x^{-2/3}, \qquad \frac{c_\alpha}{k_\alpha} x^{-2/3}, \tag{B.34}$$

are independent of rapidity and only the following combinations of relations in the subgroups $3A, 3B, 3C$ are permitted, namely

$$\frac{a_{\alpha+\alpha_1}}{a_\alpha} = q, \quad \frac{a_{\alpha+\alpha_0}}{a_\alpha} = \frac{1}{q}, \quad \frac{b_{\alpha+\alpha_1}}{b_\alpha} = \frac{1}{q}, \quad \frac{b_{\alpha+\alpha_0}}{b_\alpha} = q, \quad \frac{c_{\alpha+\alpha_0}}{c_\alpha} = q, \quad \frac{c_{\alpha+\alpha_2}}{c_\alpha} = \frac{1}{q},$$

with

$$\frac{h_w(\alpha_1)}{h_k(\alpha_0)} = \frac{h_i(\alpha_1)}{h_k(\alpha_2)} = \frac{h_i(\alpha_0)}{h_w(\alpha_2)} = 1.$$

Finally, the relations in group 5 disappear completely.

The two 'minimal' solutions of this type are

$$\begin{pmatrix} Q^{\alpha \cdot l_1}\, \delta_\alpha^\beta & 0 & t_{13}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_0} \\ t_{21}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_1} & t_{22}\, Q^{\alpha \cdot l_2}\, \delta_\alpha^\beta & 0 \\ 0 & t_{32}\, x^{2/3}\, \delta_\alpha^{\beta+\alpha_2} & t_{33}\, Q^{\alpha \cdot l_3}\, \delta_\alpha^\beta \end{pmatrix} \tag{B.35}$$

and

$$\begin{pmatrix} Q^{-\alpha\cdot l_1}\,\delta_\alpha^\beta & t_{12}\,x^{-2/3}\,\delta_\alpha^{\beta-\alpha_1} & 0 \\ 0 & t_{22}\,Q^{-\alpha\cdot l_2}\,\delta_\alpha^\beta & t_{23}\,x^{-2/3}\,\delta_\alpha^{\beta-\alpha_2} \\ t_{31}\,x^{-2/3}\,\delta_\alpha^{\beta-\alpha_0} & 0 & t_{33}\,Q^{-\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix} \tag{B.36}$$

and these are of relevance to the defect for the reasons explained earlier.

For these types of solution there are also general families. For example, (B.36) belongs to the following set

$$\begin{pmatrix} Q^{\alpha\cdot l_1}\,\delta_\alpha^\beta & 0 & t_{13}\,x^{2/3}\,Q^{-\alpha\cdot\hat{l}_2}\,\delta_\alpha^{\beta+\alpha_0} \\ t_{21}\,x^{2/3}\,Q^{-\alpha\cdot\hat{l}_3}\,\delta_\alpha^{\beta+\alpha_1} & t_{22}\,Q^{\alpha\cdot l_2}\,\delta_\alpha^\beta & 0 \\ 0 & t_{32}\,x^{2/3}\,Q^{-\alpha\cdot\hat{l}_1}\,\delta_\alpha^{\beta+\alpha_2} & t_{33}\,Q^{\alpha\cdot l_3}\,\delta_\alpha^\beta \end{pmatrix} \tag{B.37}$$

where the vectors $\hat{l}_p$ satisfy (B.32).

References

[1] G. Delfino, G. Mussardo and P. Simonetti, *Statistical models with a line of defect*, *Phys. Lett.* **B 328** (1994) 123 [hep-th/9403049]; *Scattering theory and correlation functions in statistical models with a line of defect*, *Nucl. Phys.* **B 432** (1994) 518 [hep-th/9409076].

[2] O.A. Castro-Alvaredo, A. Fring and F. Gohmann, *On the absence of simultaneous reflection and transmission in integrable impurity systems*, hep-th/0201142.

[3] M. Mintchev, E. Ragoucy and P. Sorba, *Scattering in the presence of a reflecting and transmitting impurity*, *Phys. Lett.* **B 547** (2002) 313 [hep-th/0209052].

[4] P. Bowcock, E. Corrigan and C. Zambon, *Classically integrable field theories with defects*, *Int. J. Mod. Phys.* **A 19** (2004) 82 [hep-th/0305022].

[5] P. Bowcock, E. Corrigan and C. Zambon, *Affine Toda field theories with defects*, *JHEP* **01** (2004) 056 [hep-th/0401020].

[6] E. Corrigan and C. Zambon, *Jump-defects in the nonlinear Schrödinger model and other non relativistic field theories*, *Nonlinearity* **19** (2006) 1447 [nlin.si/0512038].

[7] V. Caudrelier, *On a systematic approach to defects in classical integrable field theories*, arXiv:0704.2326.

[8] P. Bowcock, E. Corrigan and C. Zambon, *Some aspects of jump-defects in the quantum sine-Gordon model*, *JHEP* **08** (2005) 023 [hep-th/0506169].

[9] R. Konik and A. LeClair, *Purely transmitting defect field theories*, *Nucl. Phys.* **B 538** (1999) 587 [hep-th/9703085].

[10] T.J. Hollowood, *Solitons in affine Toda field theories*, *Nucl. Phys.* **B 384** (1992) 523.

[11] T.J. Hollowood, *Quantizing* $\mathrm{SL}(N)$ *solitons and the Hecke algebra*, *Int. J. Mod. Phys.* **A 8** (1993) 947 [hep-th/9203076].

[12] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Quantum S matrix of the* $(1+1)$*-dimensional Toda chain*, *Phys. Lett.* **B 87** (1979) 389;
A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, *Two-dimensional generalized Toda lattice*, *Commun. Math. Phys.* **79** (1981) 473;
G. Wilson, *The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras*, *Ergod. Th. and Dynam. Sys.* **1** (1981) 361.

JHEP07(2007)001

[13] W.A. McGhee, *The topological charges of the $a_n^{(1)}$ affine Toda solitons*, *Int. J. Mod. Phys.* **A 9** (1994) 2645 [hep-th/9307035].

[14] D.I. Olive, N. Turok and J.W.R. Underwood, *Affine Toda solitons and vertex operators*, *Nucl. Phys.* **B 409** (1993) 509 [hep-th/9305160];
E.J. Beggs and P.R. Johnson, *The generic soliton of the a_n affine Toda field theories*, hep-th/9511225; *Inverse scattering and solitons in $a(n-1)$ affine Toda field theories. II*, *Nucl. Phys.* **B 529** (1998) 567 [hep-th/9803248].

[15] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Affine Toda field theory and exact S matrices*, *Nucl. Phys.* **B 338** (1990) 689.

[16] E. Corrigan, *Recent developments in affine Toda quantum field theory*, hep-th/9412213.

[17] B. G. Wybourne, *Classical groups for physicists*, John Wiley and Sons (1974).

[18] M. Jimbo, *Introduction to the Yang-Baxter equation*, *Int. J. Mod. Phys.* **A 4** (1989) 3759.

[19] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253.

[20] G.M. Gandenberger, *Exact S matrices for bound states of $a_2^{(1)}$ affine Toda solitons*, *Nucl. Phys.* **B 449** (1995) 375 [hep-th/9501136].

[21] E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, *Affine Toda field theory on a half line*, *Phys. Lett.* **B 333** (1994) 83 [hep-th/9404108].